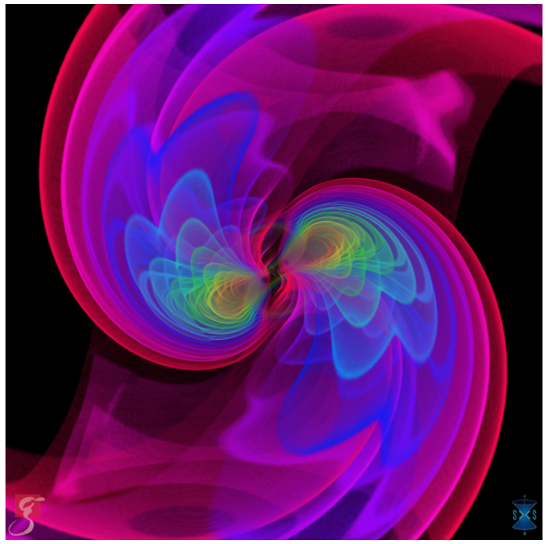

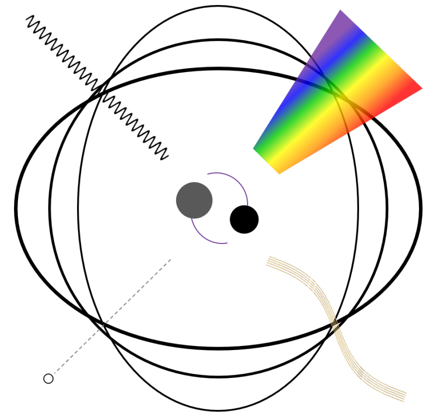

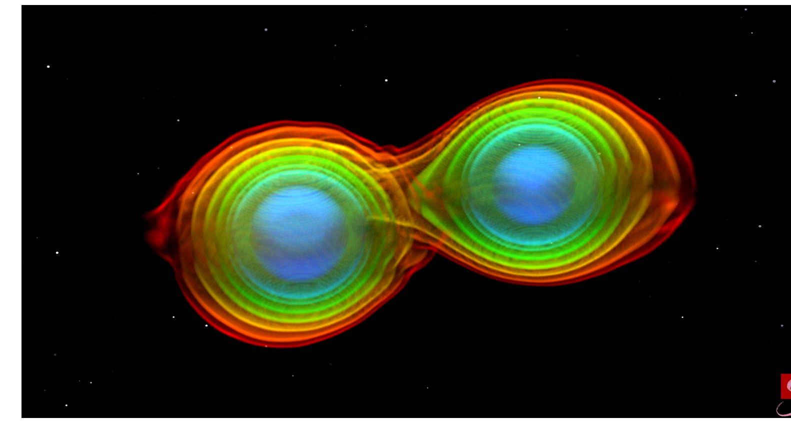

Theory/Phenomenology of GWs: Lecture 2

Alessandra Buonanno

Max Planck Institute for Gravitational Physics

(Albert Einstein Institute)

Deutsche Forschungsgemeinschaft
German Research Foundation

MAX-PLANCK-GESELLSCHAFT

University of Maryland 1856

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Outline

- **Basics of gravitational-wave modeling:** why does it matter to predict the shape of gravitational waves from binary systems?

- Waveform models in **post-Newtonian theory** for **inspiraling binary systems**, their **range of validity**, and their **use in LIGO-Virgo template banks**.

- Motivations and development of the **effective-one-body formalism** for **inspiral-merger-ringdown waveforms of binary black holes**, their **completion using numerical-relativity** waveforms, and their **use in LIGO-Virgo template banks and inference studies.**

- **Inspiral-merger-ringdown phenomenological waveforms** and their **use in LIGO-Virgo inference studies.**

- LIGO-Virgo's **science highlights** enabled by waveform models.

References

- **M. Maggiore's books:** "Gravitational Waves Volume 1: Theory and Experiments" (2007) & "Gravitational Waves Volume II: Astrophysics and Cosmology" (2018).

- **E. Poisson & C. Will's book:** "Gravity" (2015).

- **E.E. Flanagan & S.A. Hughes' review:** arXiv:0501041.

- **AB's Les Houches School Proceedings:** arXiv:0709.4682.

- **AB & B. Sathyaprakash's review:** arXiv:1410.7832.

- **UMD/AEI graduate course on GW Physics & Astrophysics** taught in Winter-Spring 2017 *(https://www.aei.mpg.de/139568/phy879)* and **HU/AEI master course on GWs** taught in Winter 2020-2021 *(https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/)*.

The effective-one-body formalism

Can we get insights on accuracy from PN 2-body Hamiltonian?

(Damour, Jaranowski & Schäfer 16)

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{Gm_1m_2}{2r_{12}} + (1\leftrightarrow 2),$$

$$H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{(\mathbf{p}_1^2)^2}{8m_1^3} + \frac{Gm_1m_2}{4r_{12}}\left(-6\frac{\mathbf{p}_1^2}{m_1^2} + 7\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) + \frac{G^2m_1^2m_2}{2r_{12}^2} + (1\leftrightarrow 2),$$

$$\begin{aligned}
H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)\\
&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) - \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2} + (1\leftrightarrow 2).
\end{aligned}$$

$$\begin{aligned}
H_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a) = {}& -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2} - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}\right.\\
&+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\
&\left.+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right) + \frac{G^2m_1m_2}{r_{12}^2}\left(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
&-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
&\left.-\frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right) + \frac{G^3m_1m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1m_2 + 150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
&+\frac{1}{16}\left(77(m_1^2+m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(20m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) + \frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1\leftrightarrow 2).
\end{aligned}$$

Can we get insights on accuracy from PN 2-body Hamiltonian?

$$
\begin{aligned}
H^{\mathrm{local}}_{\mathrm{4PN}}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}}\Bigg(\frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
& + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
& + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} \\
& - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}\Bigg) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 \Bigg(\frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} \\
& - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
& - \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} \\
& - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} \\
& - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} \\
& + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}\Bigg) + \frac{G^3 m_1 m_2}{r_{12}^3}\Bigg(m_1^2\bigg(\frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} \\
& - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
& + \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}\bigg) + m_1 m_2\bigg(\Big(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\Big)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \Big(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \Big(\frac{375\pi^2}{8192} - \frac{23533}{1280}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
& + \Big(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\Big)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \Big(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\Big)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \Big(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \Big(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
& - \Big(\frac{30383}{960} + \frac{36405\pi^2}{16384}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \Big(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\Big)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} + \Big(\frac{2369}{60} + \frac{35655\pi^2}{16384}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \Big(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
& + \Big(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}\bigg)\Bigg) + \frac{G^4 m_1 m_2}{r_{12}^4}\Bigg(m_1^3\bigg(\frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}\bigg) + m_1^2 m_2\bigg(\Big(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\Big)\frac{\mathbf{p}_1^2}{m_1^2} \\
& + \Big(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\Big)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \Big(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\Big)\frac{\mathbf{p}_2^2}{m_2^2} + \Big(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \Big(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} + \Big(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\Big)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}\bigg)\Bigg) \\
& + \frac{G^5 m_1 m_2}{r_{12}^5}\Bigg(-\frac{m_1^4}{16} + \Big(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\Big)m_1^3 m_2 + \Big(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\Big)m_1^2 m_2^2\Bigg) + (1 \leftrightarrow 2).
\end{aligned}
$$

The problem of motion in Newtonian gravity

- **Two-body Hamiltonian** $H_{\rm Newt} = \frac{1}{2m_1}\,\mathbf{p}_1^2 + \frac{1}{2m_2}\,\mathbf{p}_2^2 + U(r)$ $\quad U(r) = -\frac{m_1\,m_2}{r}$

- Reduction to **one-body Hamiltonian**

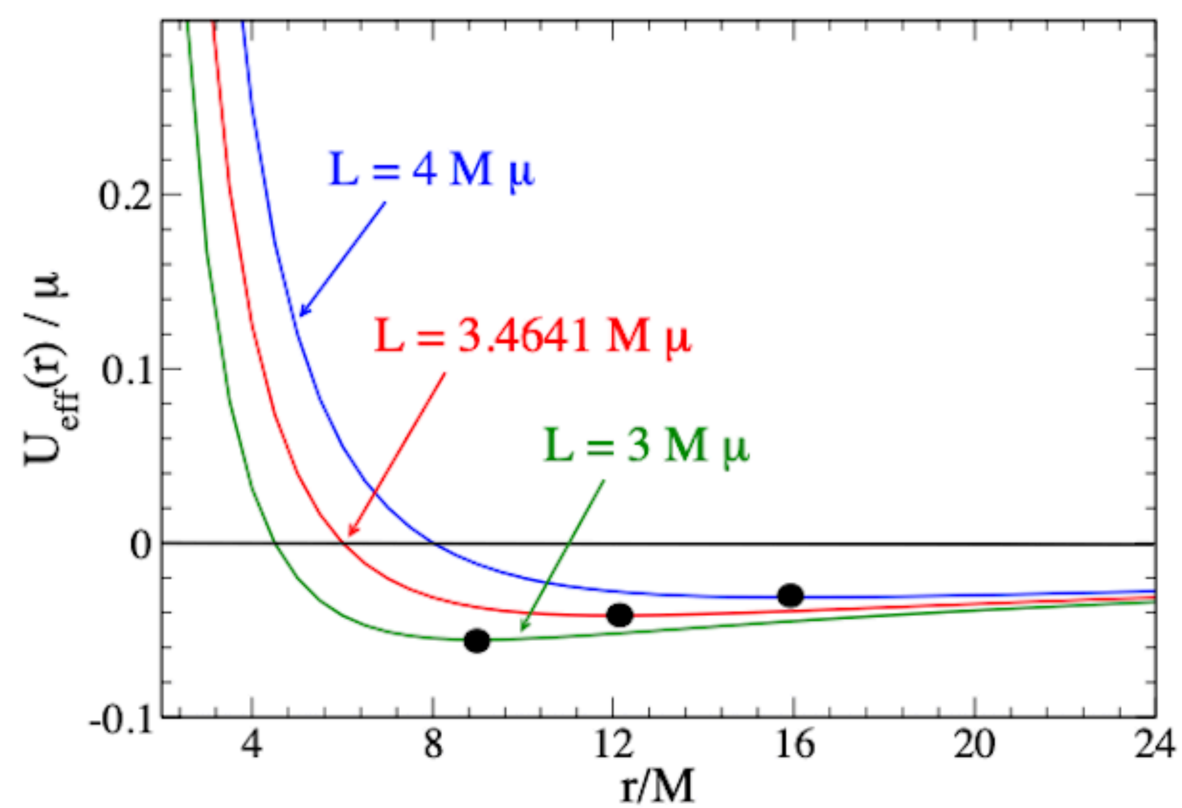
$$\mathbf{r}_{\rm CM} = \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{M} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad M = m_1 + m_2 \qquad \mu = m_1\,m_2/M$$

$$H_{\rm Newt} = \frac{1}{2\mu}\mathbf{p}^2 + U(r)$$

- **$H_{\rm Newt}(\mathbf{r},\mathbf{p})$ describes a test-body of mass μ orbiting an external mass M**

- **Effective radial potential**

$$\frac{U_{\rm eff}(r)}{\mu} = \frac{1}{2}\,\frac{L^2}{\mu^2\,r^2} - \frac{M}{r}$$

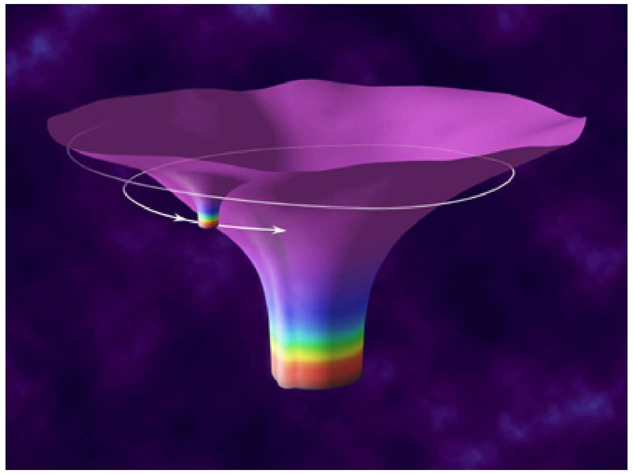


- **Bound orbits** are **closed**.

- **For any** angular momentum $L \neq 0$ **there exists a circular orbit**

One-body problem: test-particle orbiting non-spinning BH

- **Schwarzschild metric**



$$ds^2 = -\left(1 - \frac{2M}{r}\right)\, dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}\, dr^2 + r^2\, d\Omega^2$$

$$g^{\mu\nu}_{\rm Schw}\, p_\mu\, p_\nu + \mu^2 = 0 \qquad E = -\,p_0$$

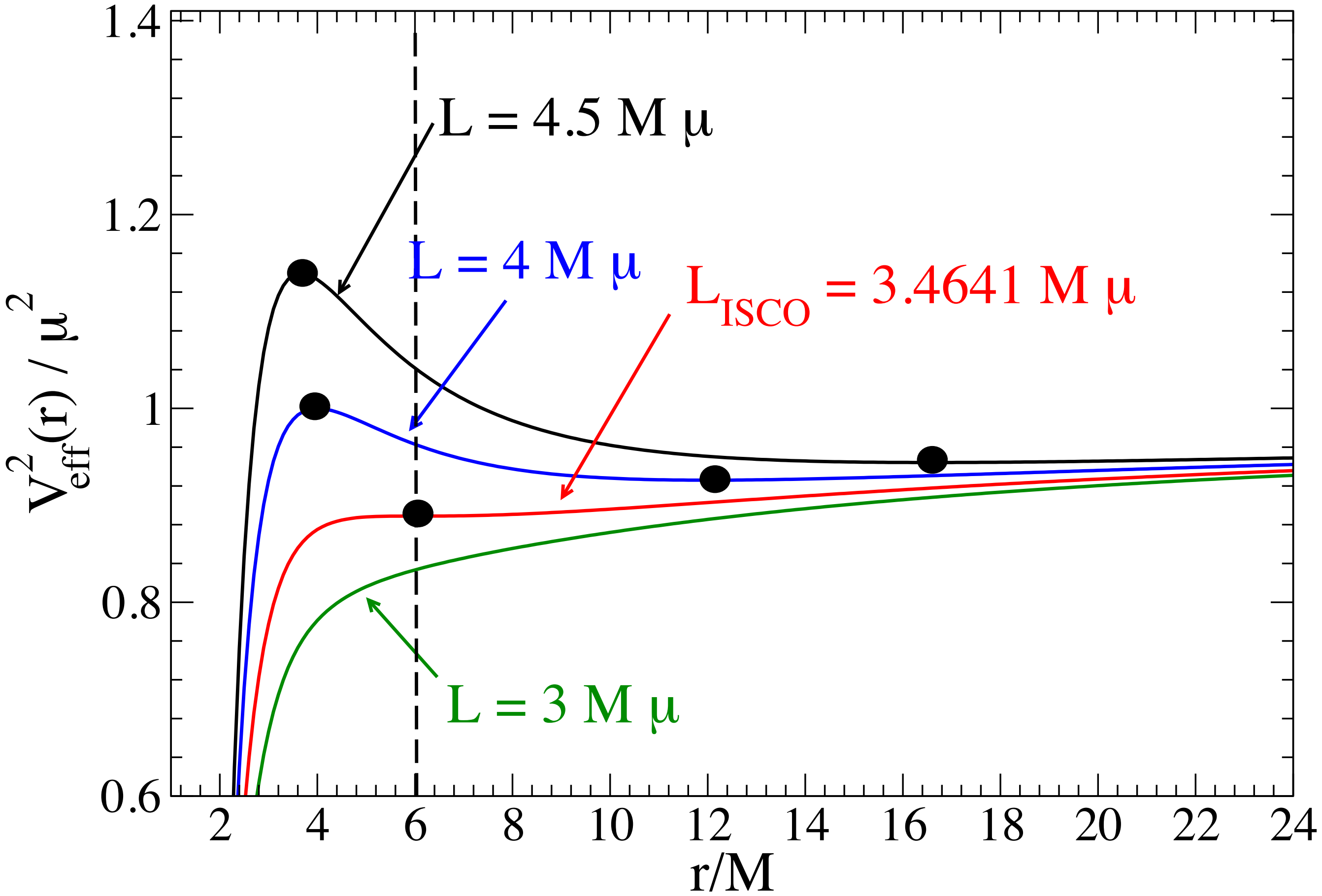
$$H_{\rm Schw}(\mathbf{r},\mathbf{p}) = \mu \sqrt{\left(1 - \frac{2M}{r}\right)\, \left[1 + \frac{\mathbf{p}^2}{\mu^2} - \frac{2M}{r}\, \frac{p_r^2}{\mu^2}\right]}$$

- $H_{\rm Schw}$ describes a **test-body of mass μ** orbiting a black hole of mass M.

- **Effective radial potential**

$$\frac{V^2_{\rm eff}(r)}{\mu^2} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{\mu^2\, r^2}\right)$$

- For L < L$_{\rm ISCO}$ **circular orbits no longer** exist.



How far in strong-field regime can we push PN approximation?

- Circular-orbit **energy for a test-body in Schwarzschild** spacetime:

$$g^{\mu\nu}_{\rm Schw}\, p_\mu\, p_\nu + \mu^2 = 0 \qquad\qquad E = -\,p_0$$

$$E_{\rm circ}(r) = \mu\, \frac{1 - 2M/r}{\sqrt{1-3M/r}} \qquad\qquad E^{\rm PN}_{\rm circ}(r) = \mu - \frac{\mu\, M}{2r}\left(1 - \frac{3}{4}\,\frac{M}{r} - \frac{27}{8}\,\frac{M^2}{r^2} - \frac{675}{64}\,\frac{M^3}{r^3} + \cdots\right)$$

- **Minimum of $E_{\rm circ}$** gives the ISCO: $r_{\rm ISCO} = 6\,M$

$E^{\rm 1PN}_{\rm circ} \Rightarrow r^{\rm 1PN}_{\rm ISCO} = 1.5\,M$

$E^{\rm 2PN}_{\rm circ} \Rightarrow r^{\rm 2PN}_{\rm ISCO} = 4.019\,M$

$E^{\rm 3PN}_{\rm circ} \Rightarrow r^{\rm 3PN}_{\rm ISCO} = 5.104\,M$

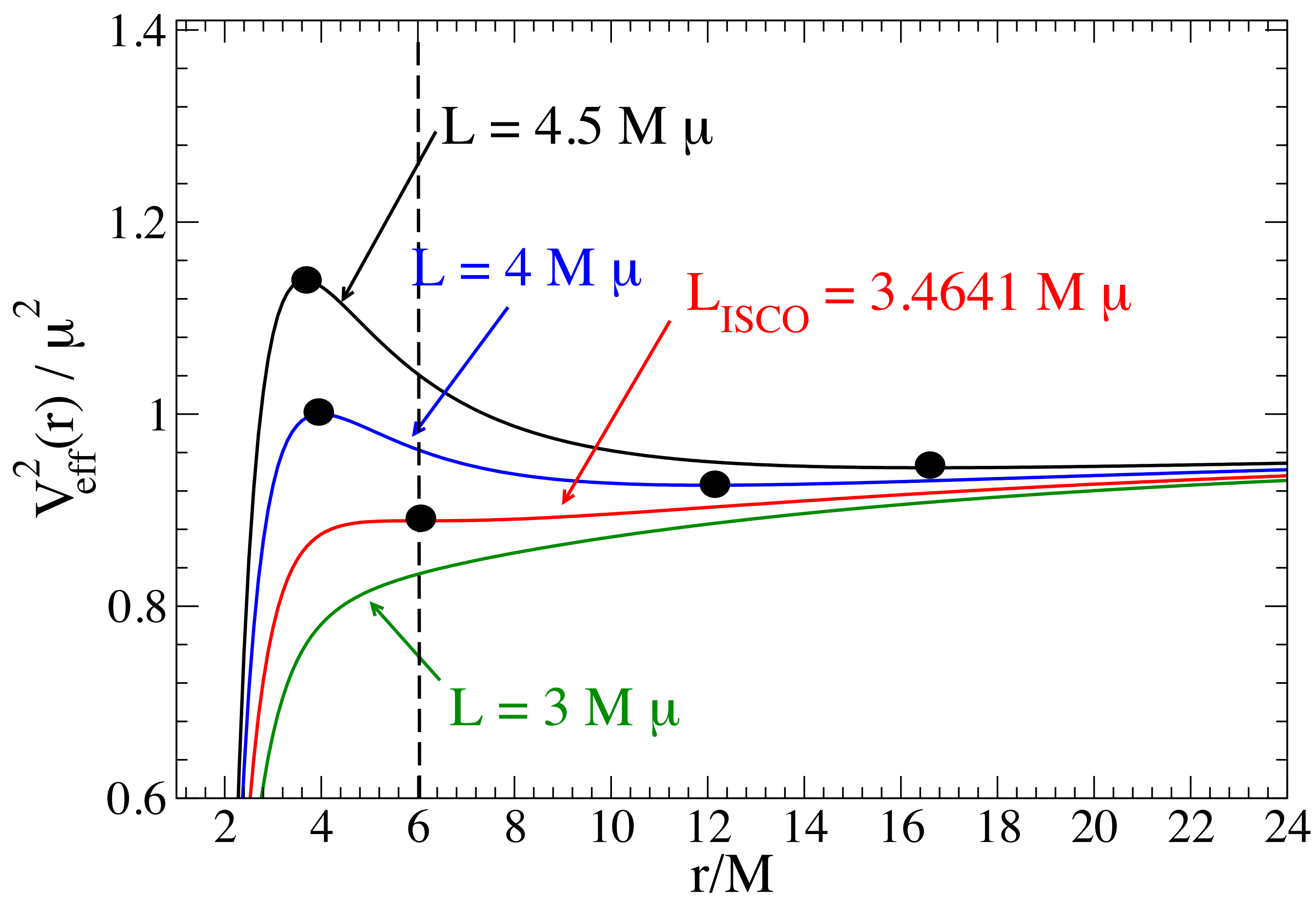
$E^{\rm 4PN}_{\rm circ} \Rightarrow r^{\rm 4PN}_{\rm ISCO} = 5.572\,M$

$E^{\rm 5PN}_{\rm circ} \Rightarrow r^{\rm 5PN}_{\rm ISCO} = 5.788\,M$

$E^{\rm 6PN}_{\rm circ} \Rightarrow r^{\rm 6PN}_{\rm ISCO} = 5.892\,M$

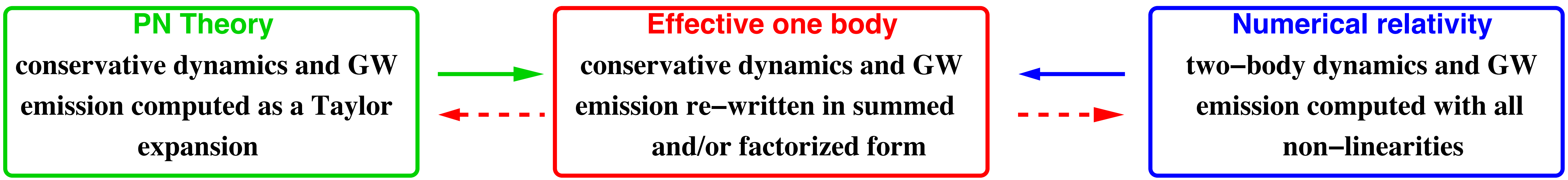
$\cdots$

$E^{\rm 10PN}_{\rm circ} \Rightarrow r^{\rm 10PN}_{\rm ISCO} = 5.992\,M$



The effective-one-body (EOB) approach

- **EOB** approach **introduced before NR** breakthrough.



- **EOB** model uses best information available in PN theory, but **resums PN terms** in suitable way to describe accurately dynamics and radiation during inspiral and plunge (going beyond quasi-circular, adiabatic motion).

- **EOB** assumes **comparable-mass** description is **smooth deformation of test-body limit**. It employs **non-perturbative** ingredients and **models analytically merger-ringdown** signal.

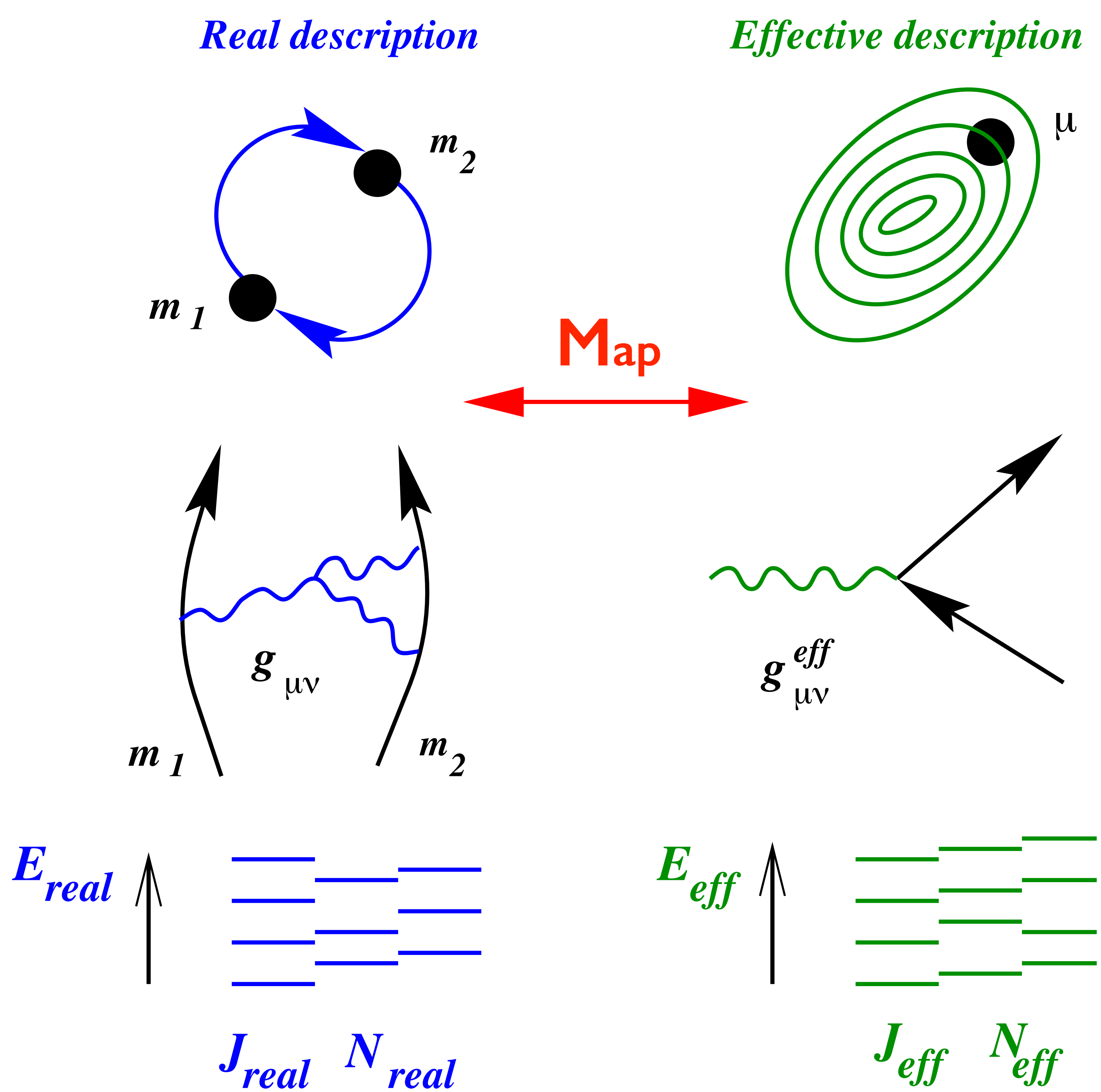
The effective-one-body approach in a nutshell

$$\nu = \frac{\mu}{M} \qquad 0 \leq \nu \leq 1/4$$

$$\mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$

- **Two-body dynamics is mapped** into dynamics of **one-effective body** moving **in deformed black-hole spacetime**, deformation being the mass ratio.

- Some key **ideas** of EOB model were **inspired by quantum** field **theory** when describing energy of comparable-mass charged bodies.



(AB & Damour 1998)

Finding the energy for comparable-mass binary black holes

- **Thinking "quantum mechanically"** (à la Wheeler): N & J **are classical action variables,** and are "quantized" in integers. Natural to require that **"quantum numbers" (N & J) between real and effective** descriptions be **the same**.

- ***Real description:***

$$E_{\rm real}(N,J) = Mc^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1+\frac{\alpha^2}{c^2}\left(\frac{6}{NJ}-\frac{1}{4}\frac{15-4\nu}{N^2}\right)+\cdots\right], \quad \alpha = GM\mu$$

- ***Effective description:***

$$E_{\rm eff}(N,J) = \mu\, c^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1+\frac{\alpha^2}{c^2}\left(\frac{C_{3,1}}{NJ}+\frac{C_{4,0}}{N^2}\right)+\cdots\right], \quad \alpha = GM\mu$$

- Allow transformation of energy axis:

$$E_{\rm eff}^{\rm NR} = E_{\rm real}^{\rm NR}\left[1+\alpha_1\frac{E_{\rm real}^{\rm NR}}{\mu c^2}+\alpha_2\left(\frac{E_{\rm real}^{\rm NR}}{\mu c^2}\right)^2+\cdots\right]$$

Energy for comparable-mass bodies

- Classical gravity *(AB & Damour 1998):*

$$E_{\rm real}^2 = m_1^2 + m_2^2 + 2 m_1 m_2 \left(\frac{E_{\rm eff}}{\mu} \right)$$

- Quantum electrodynamics *(Brezin, Itzykson & Zinn-Justin 1970):*

$$E_{\rm real}^2 = m_1^2 + m_2^2 + 2 m_1 m_2 \frac{1}{\sqrt{1 + Z^2\, \alpha^2 / (n - \epsilon_j)^2}}$$

EOB Hamiltonian: resummed conservative dynamics

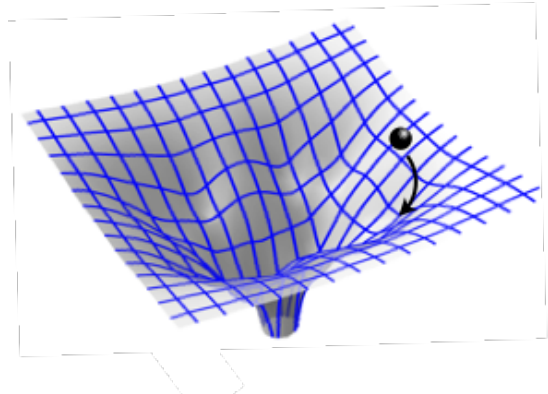
- Real Hamiltonian $\longrightarrow$ • Effective Hamiltonian

$$H_{\rm real}^{\rm PN} = H_{\rm Newt} + H_{\rm 1PN} + H_{\rm 2PN} + \cdots$$

$$H_{\rm eff}^{\nu} = \mu \sqrt{A_\nu(r)\left[1 + \frac{\mathbf{p}^2}{\mu^2} + \left(\frac{1}{B_\nu(r)} - 1\right)\frac{p_r^2}{\mu^2}\right]}$$

$$ds^2_{\rm eff} = -A_\nu(r)dt^2 + B_\nu(r)dr^2 + r^2 d\Omega^2$$

$$B_\nu(r) \equiv D_\nu(r)/A_\nu(r)$$



- EOB Hamiltonian: $H_{\rm real}^{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{{H_{\rm eff}}^{\nu}}{\mu} - 1\right)}$

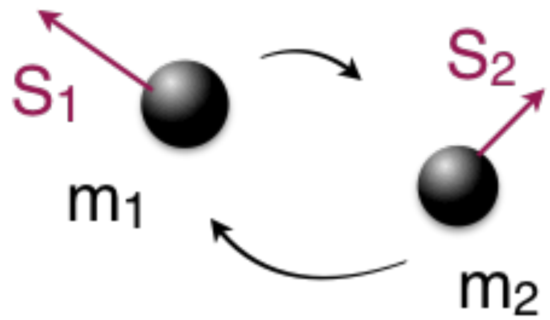
- Dynamics condensed $A_\nu(r)$ and $B_\nu(r)$

- $A_\nu(r)$, which encodes the energetics of circular orbits, is quite simple:

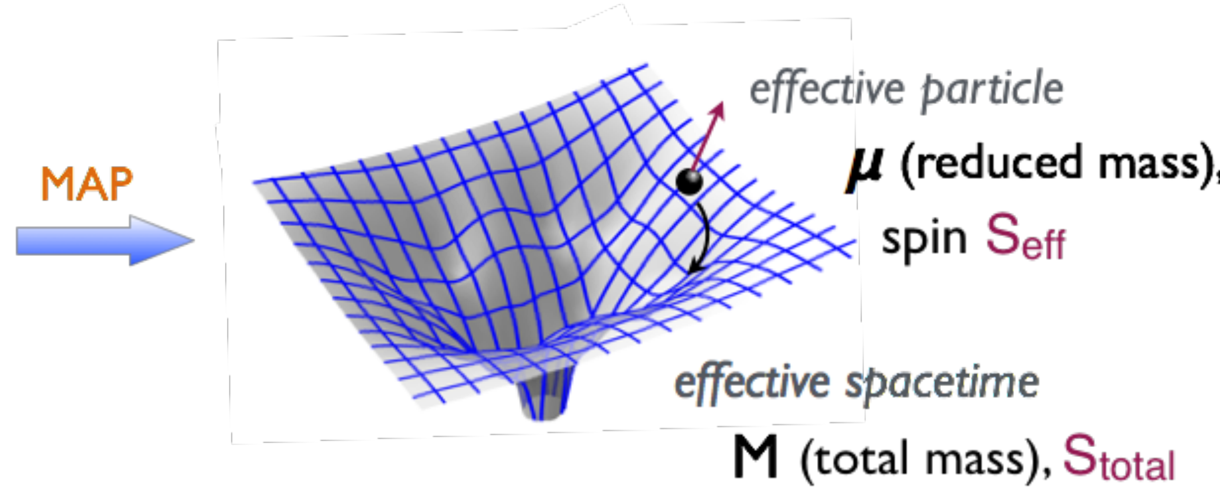
$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu) + a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

5PN

EOB dynamics and waveforms



(credit: Hinderer)



$$H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}^{\nu}}{\mu}-1\right)}$$

- EOB equations of motion *(AB et al. 00, 05; Damour et al. 09)*:

$$\dot{\mathbf{r}} = \frac{\partial H_{\rm real}^{\rm EOB}}{\partial \mathbf{p}} \qquad F \propto \frac{dE}{dt}, \quad \frac{dE}{dt} \propto \sum_{\ell m} |h_{\ell m}|^2$$

$$\dot{\mathbf{p}} = -\frac{\partial H_{\rm real}^{\rm EOB}}{\partial \mathbf{r}} + \mathbf{F} \qquad \dot{\mathbf{S}} = \{\mathbf{S}, H_{\rm real}^{\rm EOB}\}$$

- EOB waveforms *(AB et al. 00; Damour et al. 09; Pan et al. 11)*:

$$h_{\ell m}^{\rm insp-plunge} = h_{\ell m}^{\rm Newt}\, e^{-im\Phi}\, S_{\rm eff}\, T_{\ell m}\, e^{i\delta_{\ell m}}\, (\rho_{\ell m})^{\ell}\, h_{\ell m}^{\rm NQC}$$

Black-hole perturbation theory: quasi-normal modes

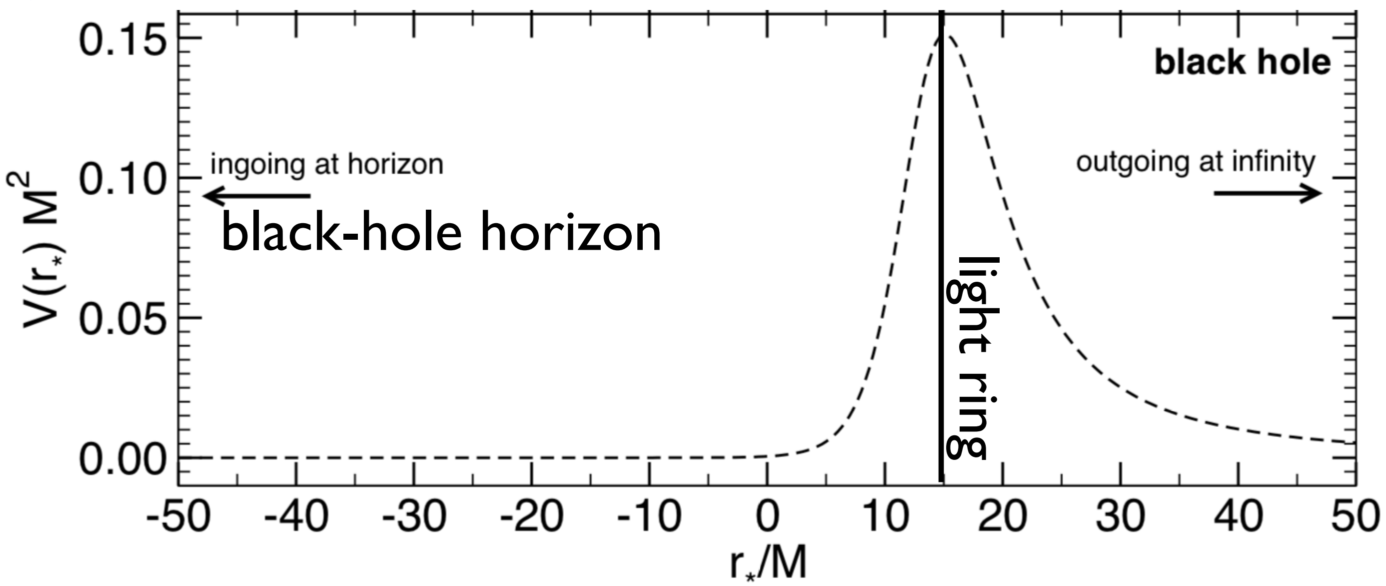
If perturbed, BHs ring or vibrate: quasi-normal modes

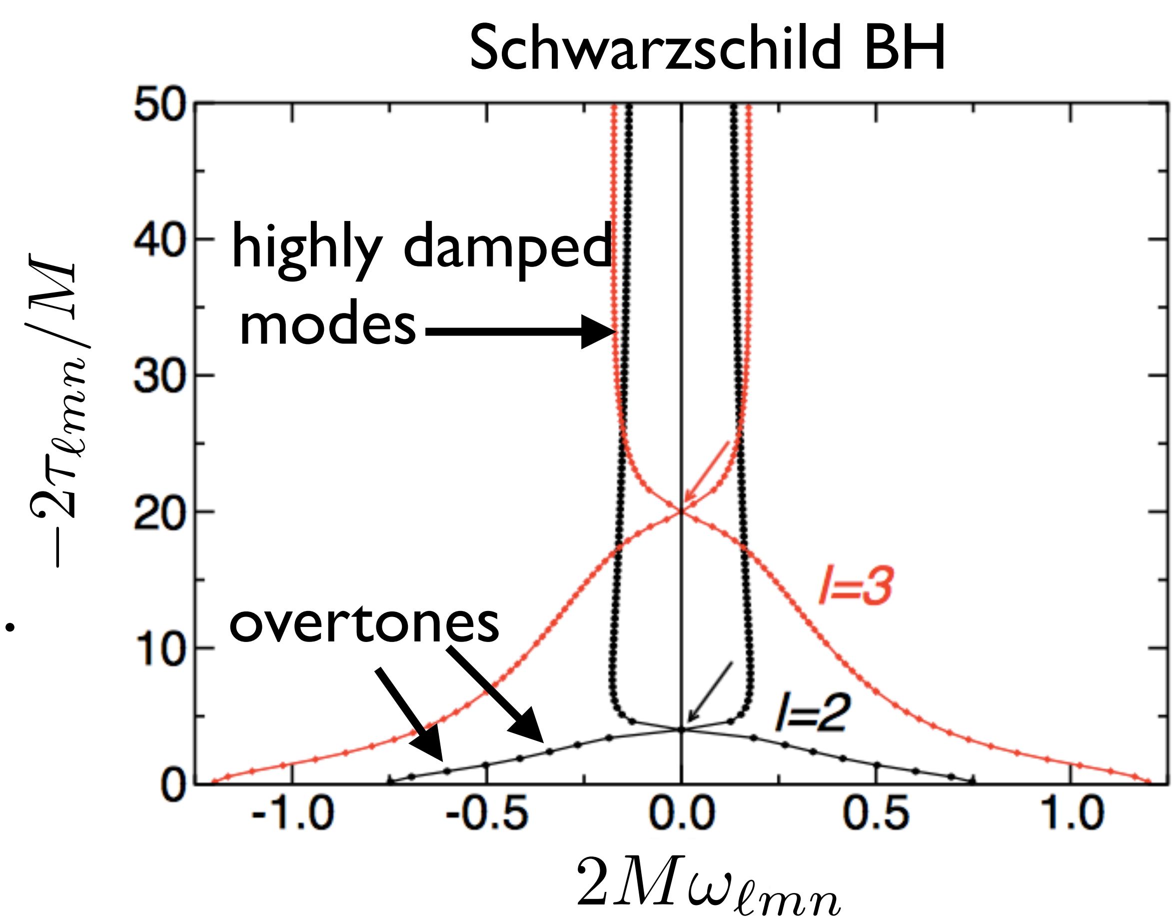
- Equation of **gravitational perturbations** in BH spacetime:

(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = 0$$

(Vishveshwara 70, Press 71, Chandrasekhar et al. 75)





- If BH's size is $R_{\rm BH} = 2GM/c^2$ and mass $M = 20 M_\odot \;\Rightarrow\; R_{\rm BH} \sim 60{\rm km}$, then the **travel time of spacetime vibration** is
  $\Rightarrow\; R_{\rm BH}/c \sim 0.2\,{\rm msec}$

- For **each (l,m)**, infinite tower of **overtones n**.

- For astrophysical black holes (zero charge), QNM's **frequency** and **decay time** only depend on **mass and spin**.

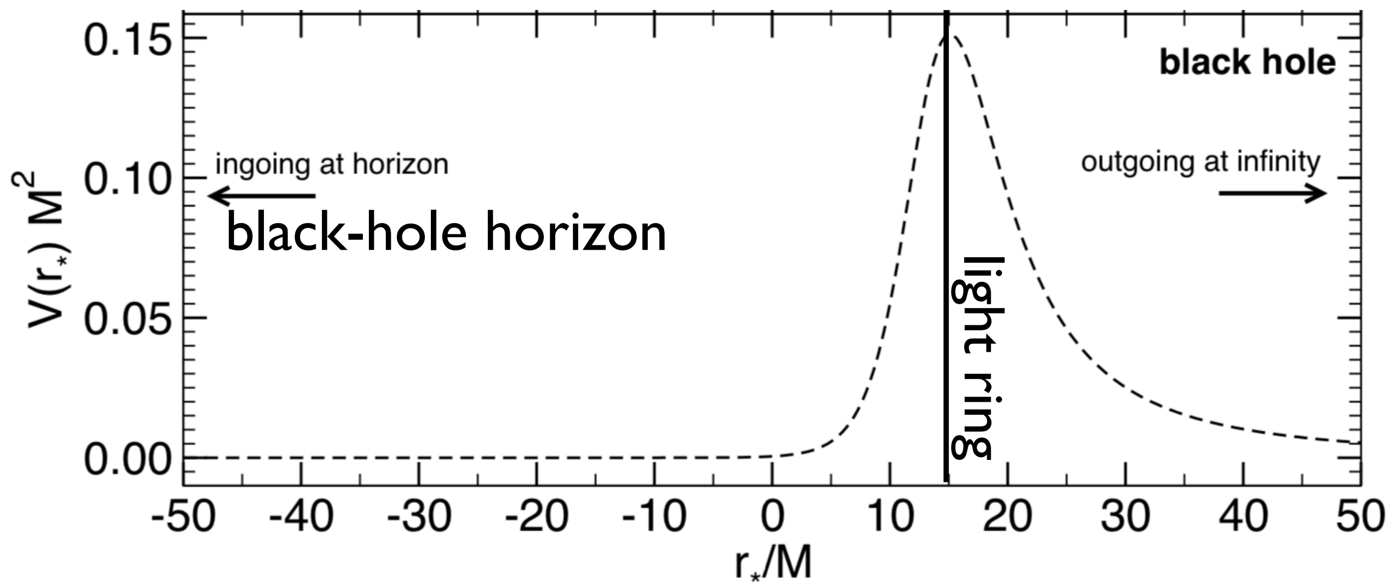
If perturbed, BHs ring or vibrate: quasi-normal modes

- Equation of **gravitational perturbations** in BH spacetime:

(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

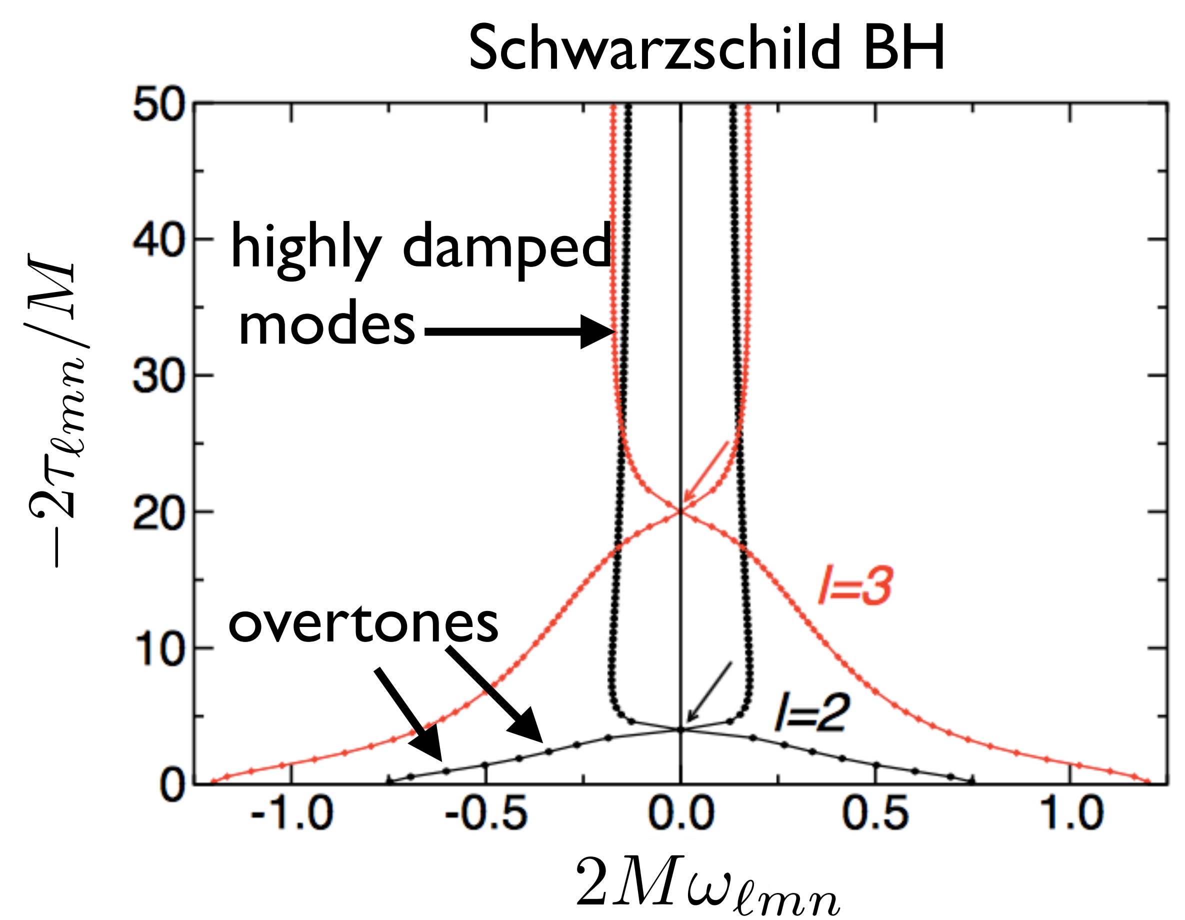
$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = 0$$

(Vishveshwara 70, Press 71, Chandrasekhar et al. 75)



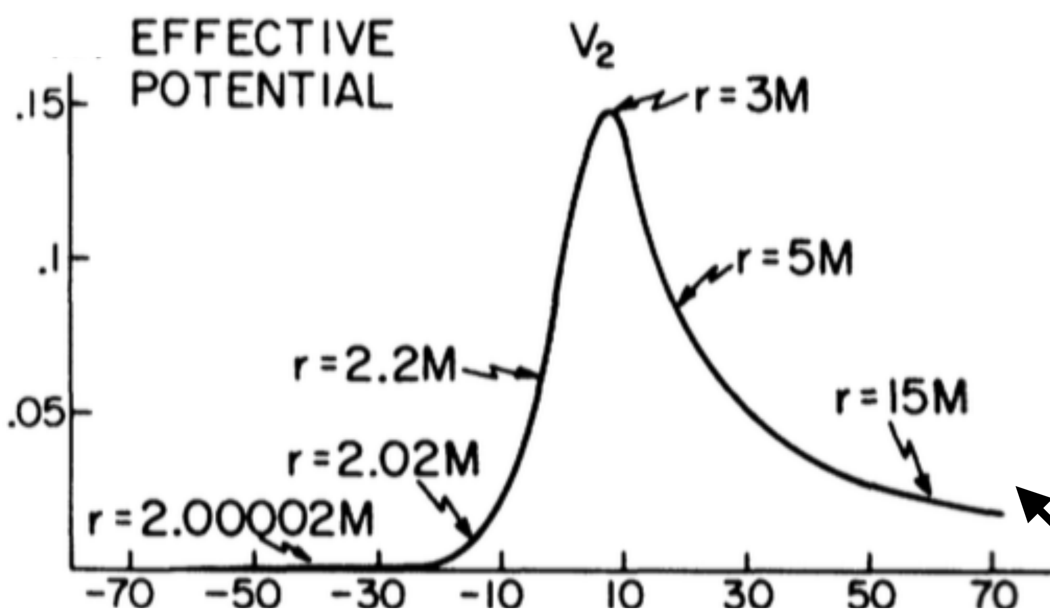
- For a Schwarzschild BH of $M = 20 M_\odot$:

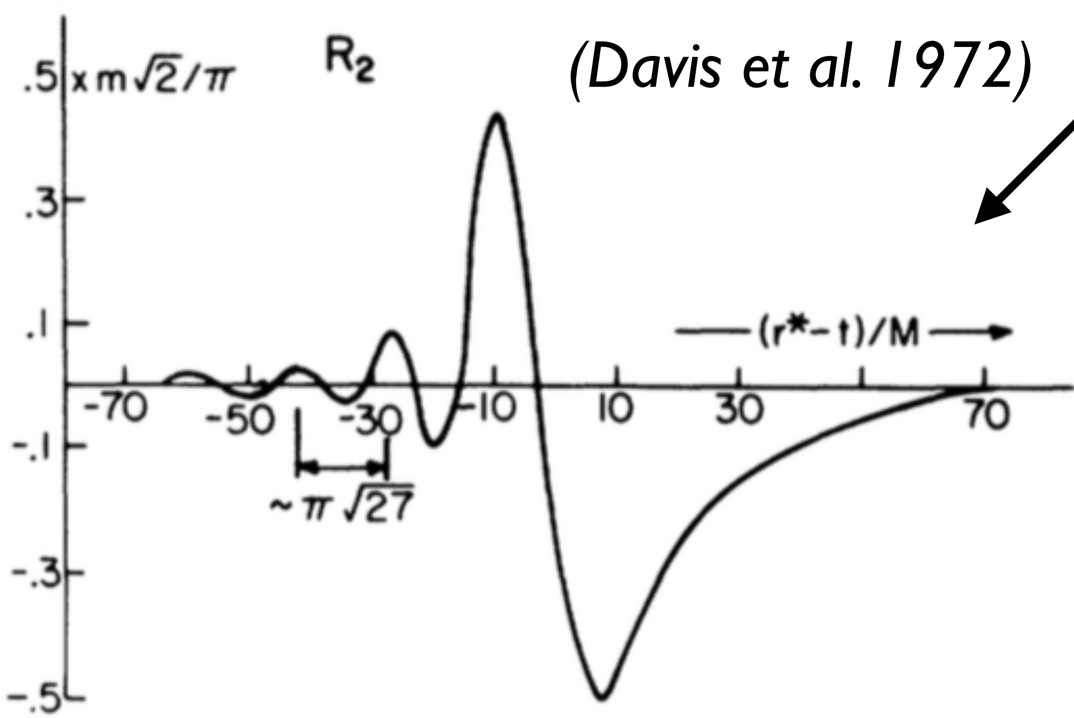
$$f_{2m0} = 604\,\mathrm{Hz},\ \tau_{2m0} = 1.10\,\mathrm{msec}$$
$$f_{2m1} = 560\,\mathrm{Hz},\ \tau_{2m1} = 0.36\,\mathrm{msec}$$
$$f_{2m2} = 486\,\mathrm{Hz},\ \tau_{2m2} = 0.20\,\mathrm{msec}$$



Merger-ringdown waveform in small-mass ratio limit

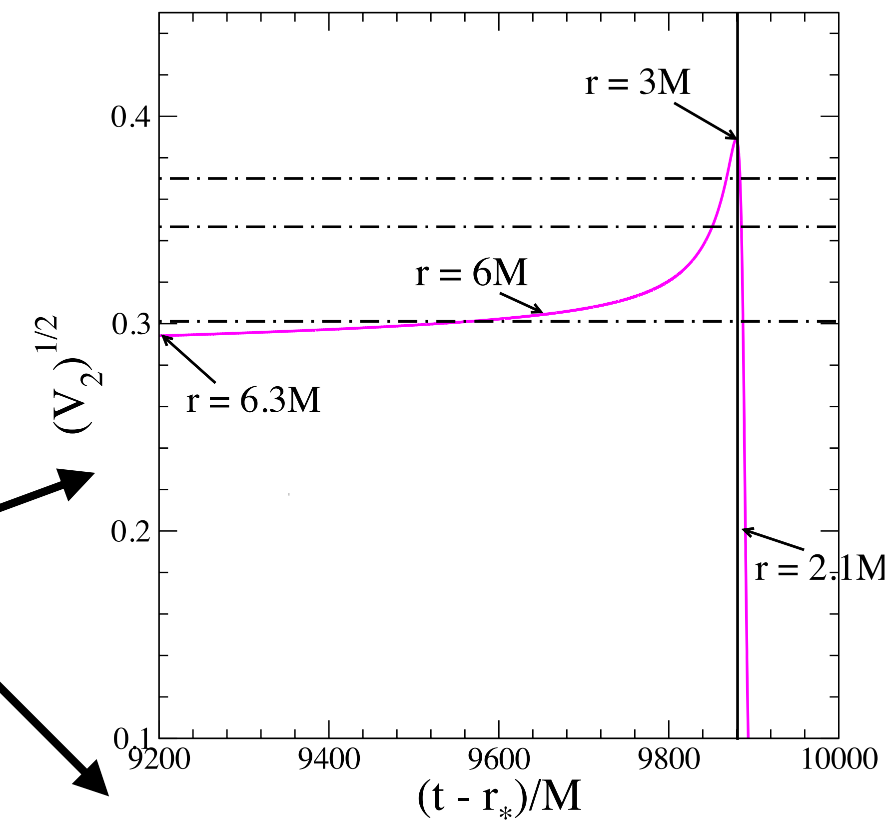
$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m}\Psi = S_{\ell m}$$

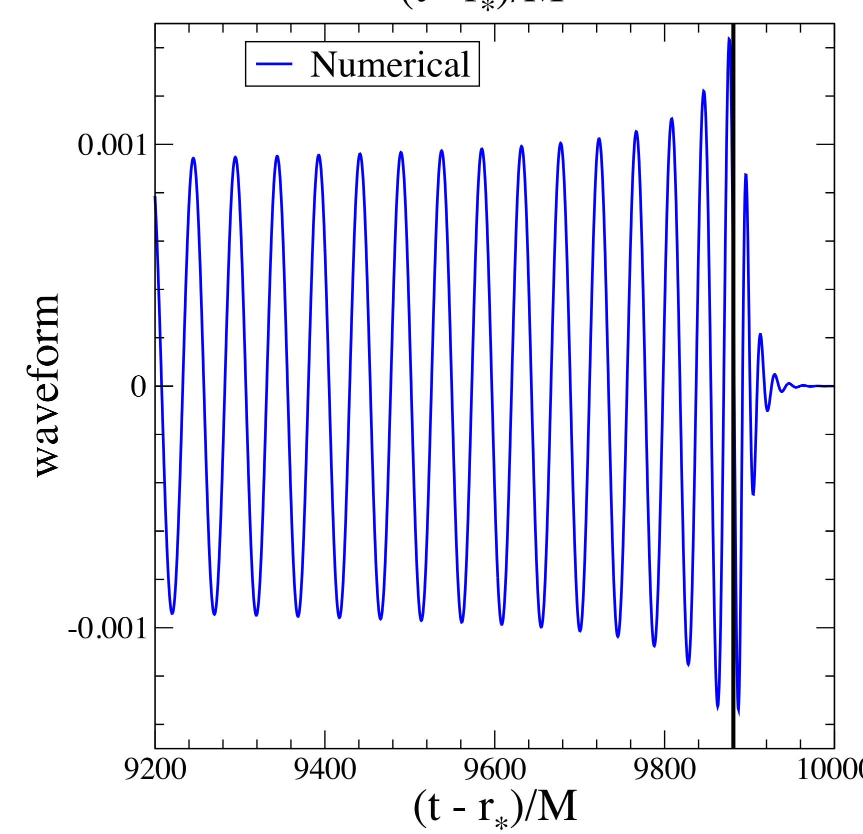




(Davis et al. 1972)

- Quasi-circular inspiral
- Radial infall





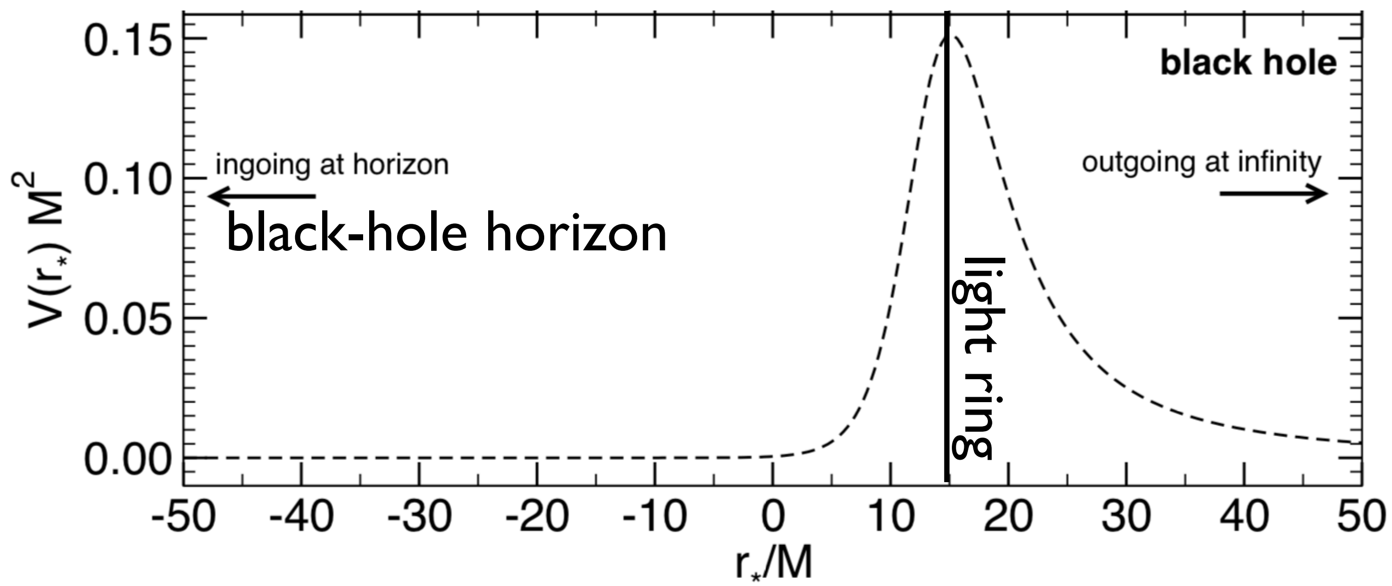
(Barausse, AB et al. 11, see also Damour & Nagar 07)

On the simplicity of merger signal in small-mass ratio limit

- Equation of gravitational perturbations in BH spacetime

(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = S_{\ell m}$$



- Peak of black-hole potential close to "light ring".

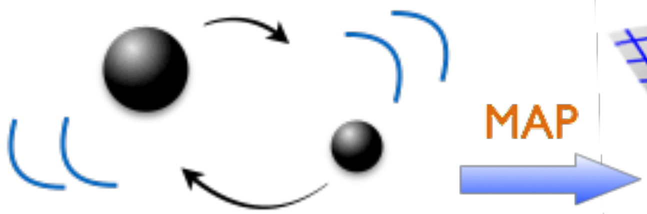
- Once particle is inside potential, direct gravitational radiation from its motion is strongly filtered by potential barrier (high-pass filter).

- Only black-hole spacetime vibrations (quasi-normal modes) leaks out BH potential.

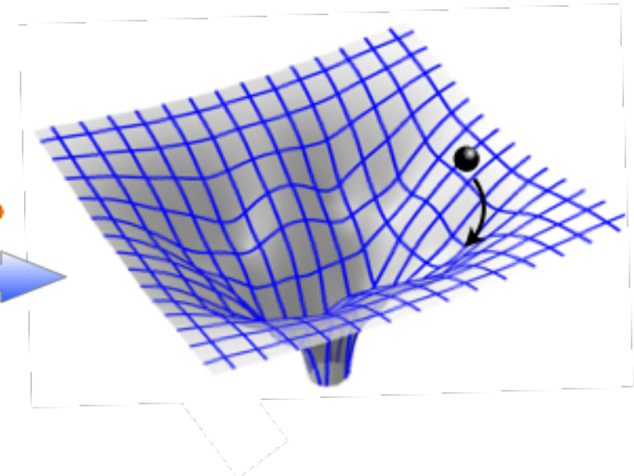
(Goebel 1972, Davis et al. 1972, Ferrari & Mashhoon 1984)

Semi-analytical estimate of the merger-ringdown signal

EOB inspiral-merger-ringdown analytic waveform



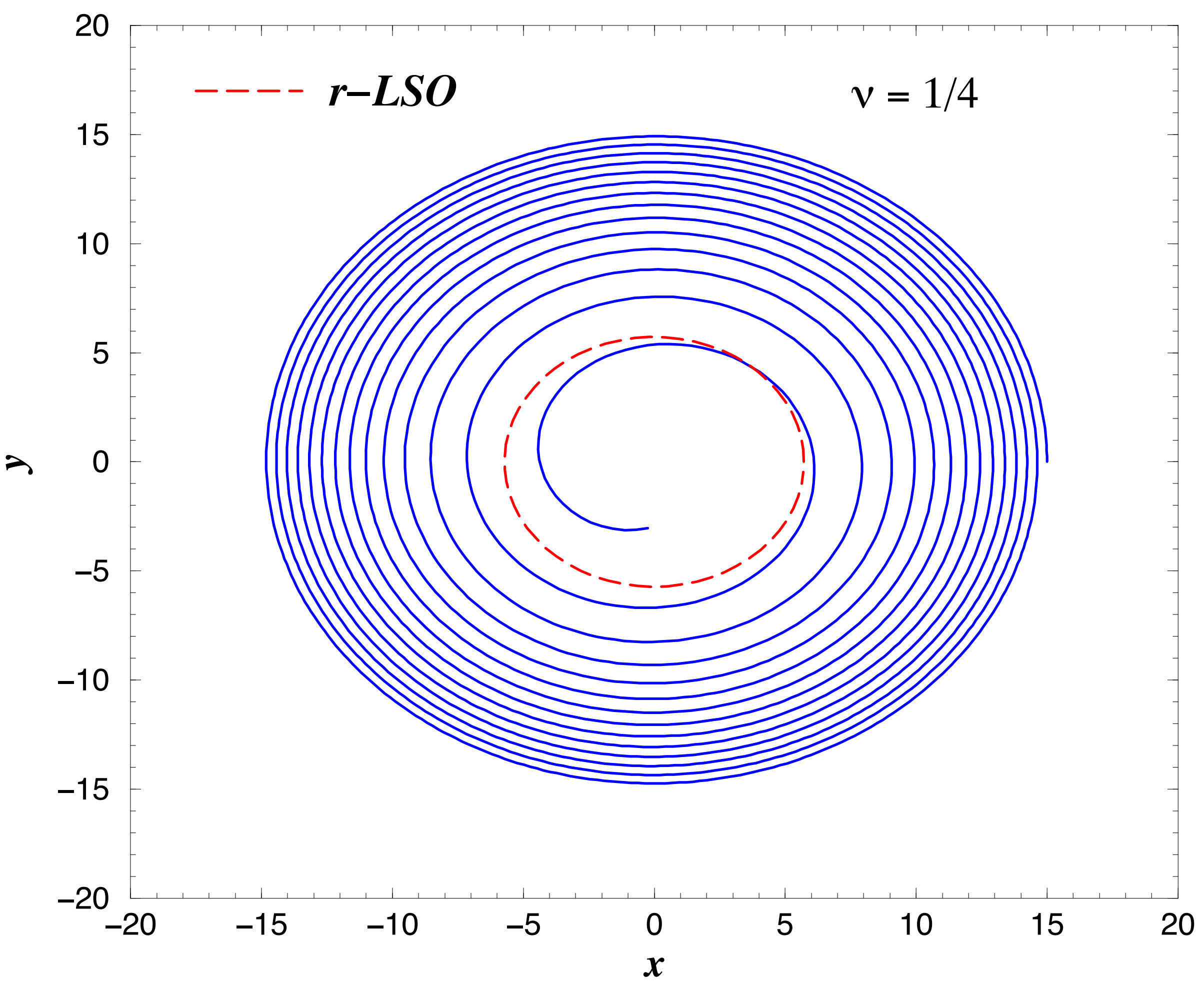
(credit: Hinderer)

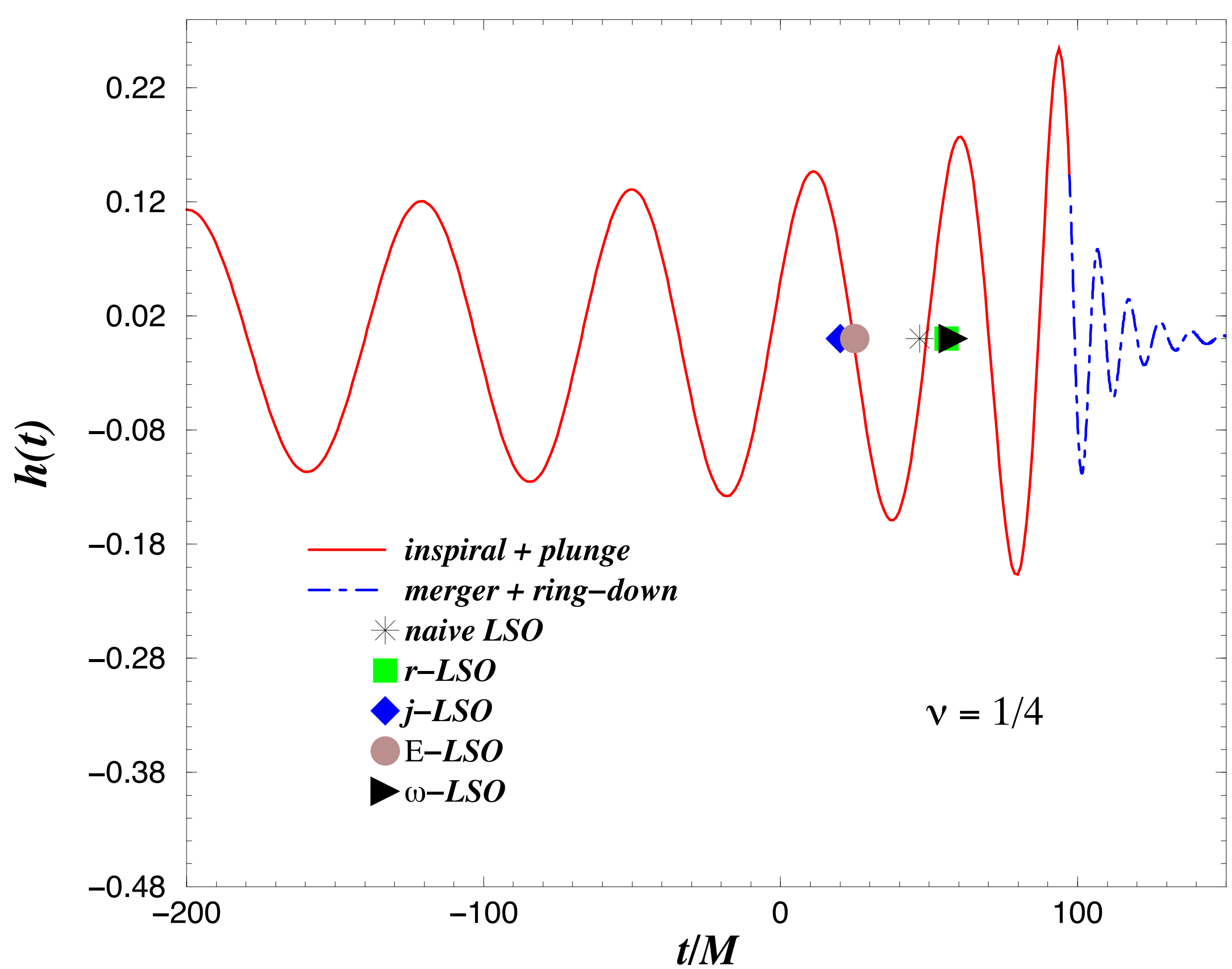


$$H_{\rm real}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \left(\frac{H_{\rm eff}}{\mu}\right)$$

$$h^{\rm insp-plunge}(t) = \nu \left(\frac{GM}{c^2 D}\right) \frac{v^2}{c^2} \cos 2\Phi$$

$$\frac{v}{c} = \left(\frac{GM\omega}{c^3}\right)^{1/3} \qquad \nu = \mu/M$$





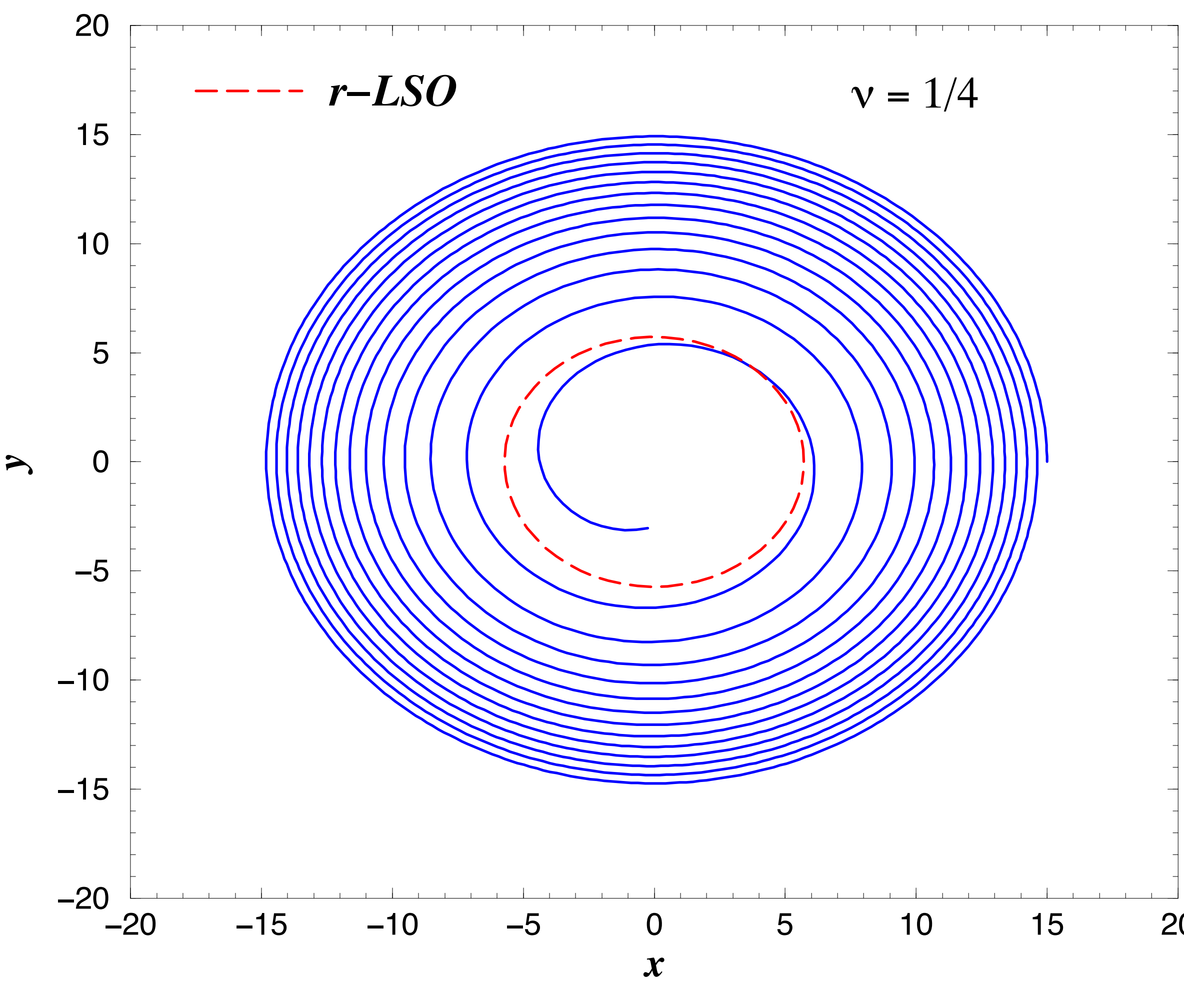
(AB & Damour 00)

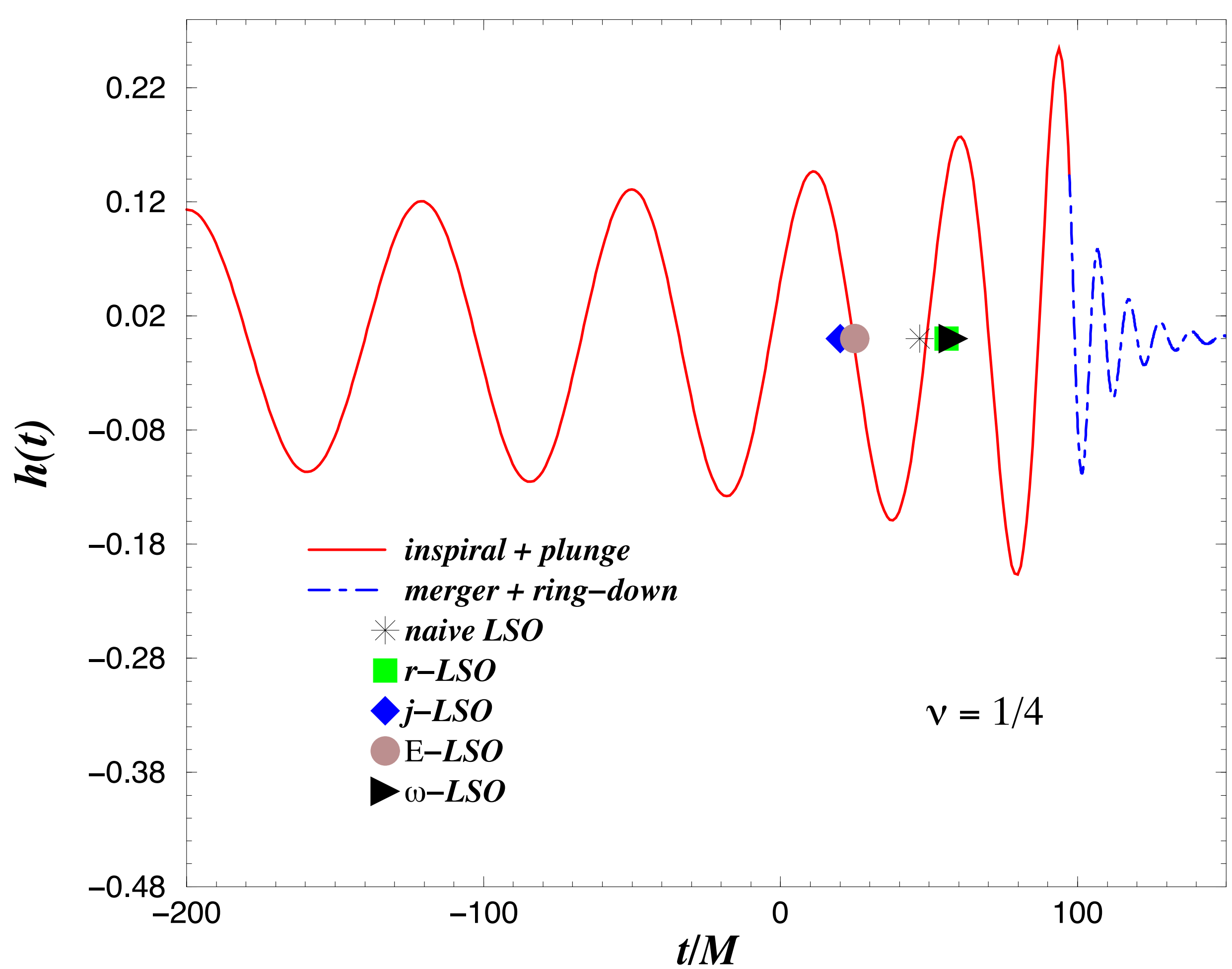
EOB inspiral-merger-ringdown analytic waveform

- The **plunge** (~ 1.5 GW cycles) is a **smooth continuation of inspiral** phase.
- The **transition merger to ringdown** is assumed to be **very short**.
- **One single QNM** is matched with $M_{\rm BH} = E_{\rm LR} = 0.976\,M,\, a_{\rm BH} = J_{\rm BH}/M_{\rm BH}^2 = 0.77$

$$h^{\rm merger-RD}(t) = A\, e^{-(t-t_{\rm match})/\tau_{\rm QNM}}\ \cos[\omega_{\rm QNM}(t-t_{\rm match})+B]$$

In 2005, NR breakthrough found 0.68 for equal-mass binary merger.

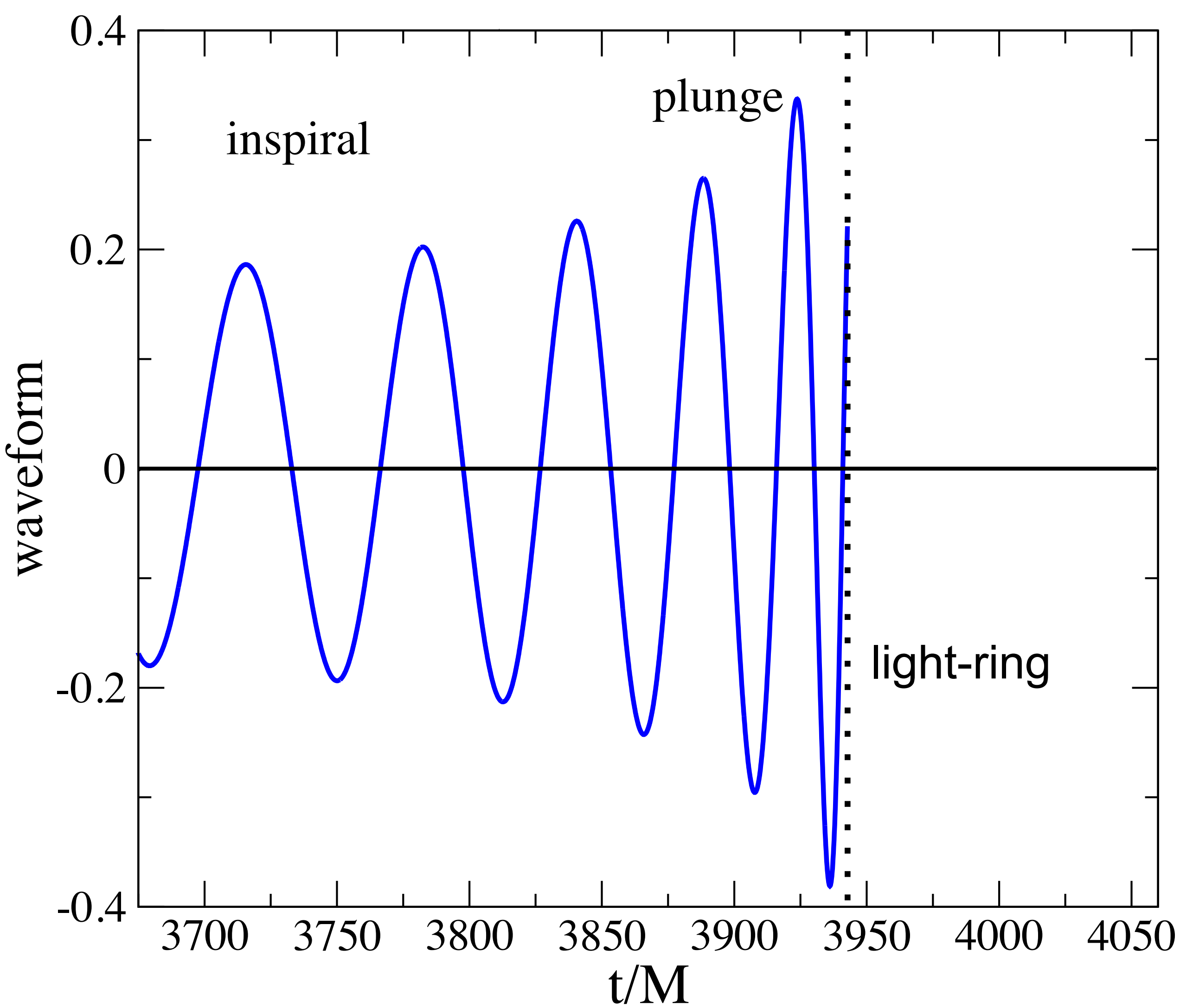


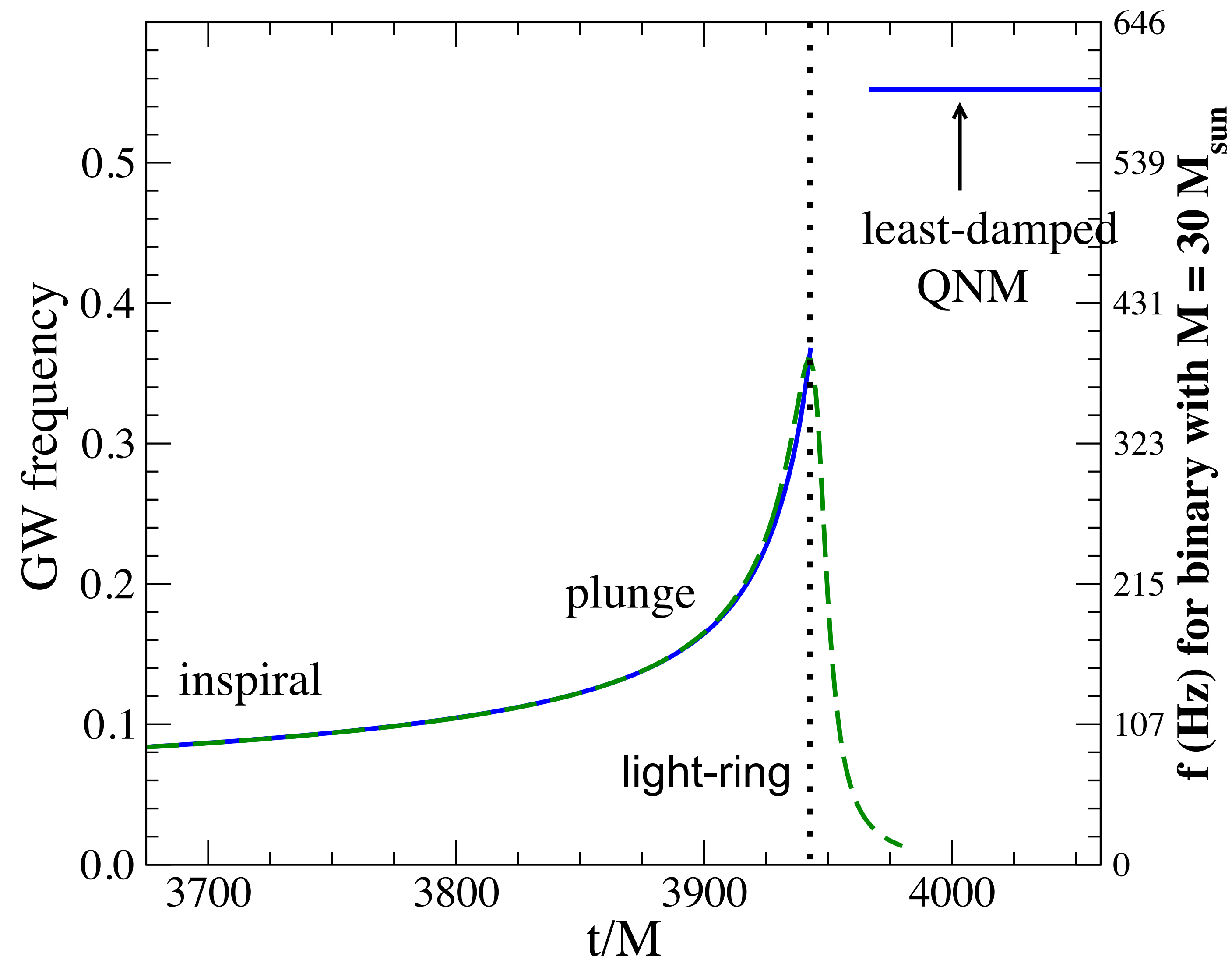


(AB & Damour 00)

EOB inspiral-plunge waveform & frequency

- Evolve **two-body dynamics up to light ring** (or photon orbit) and then …



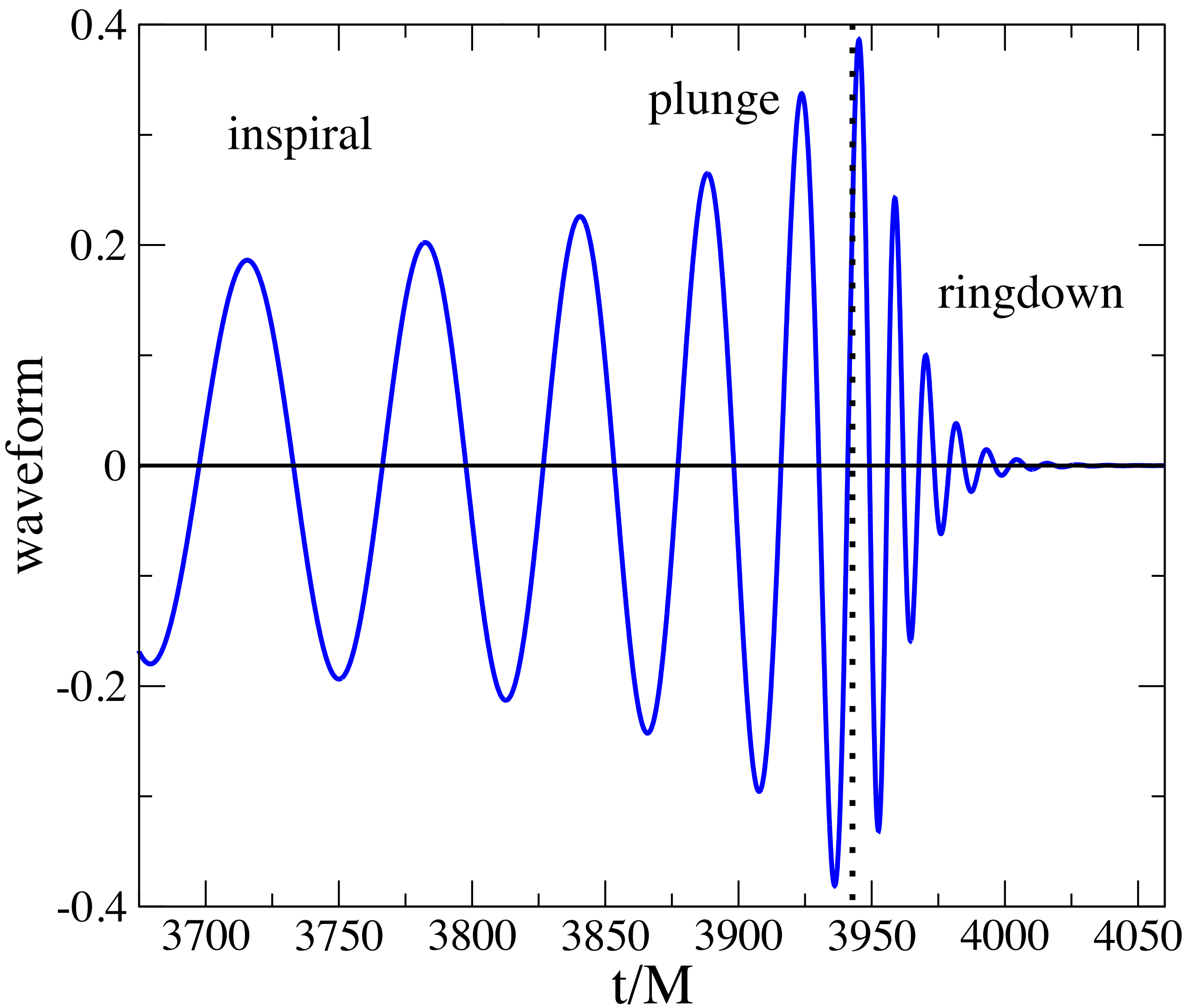


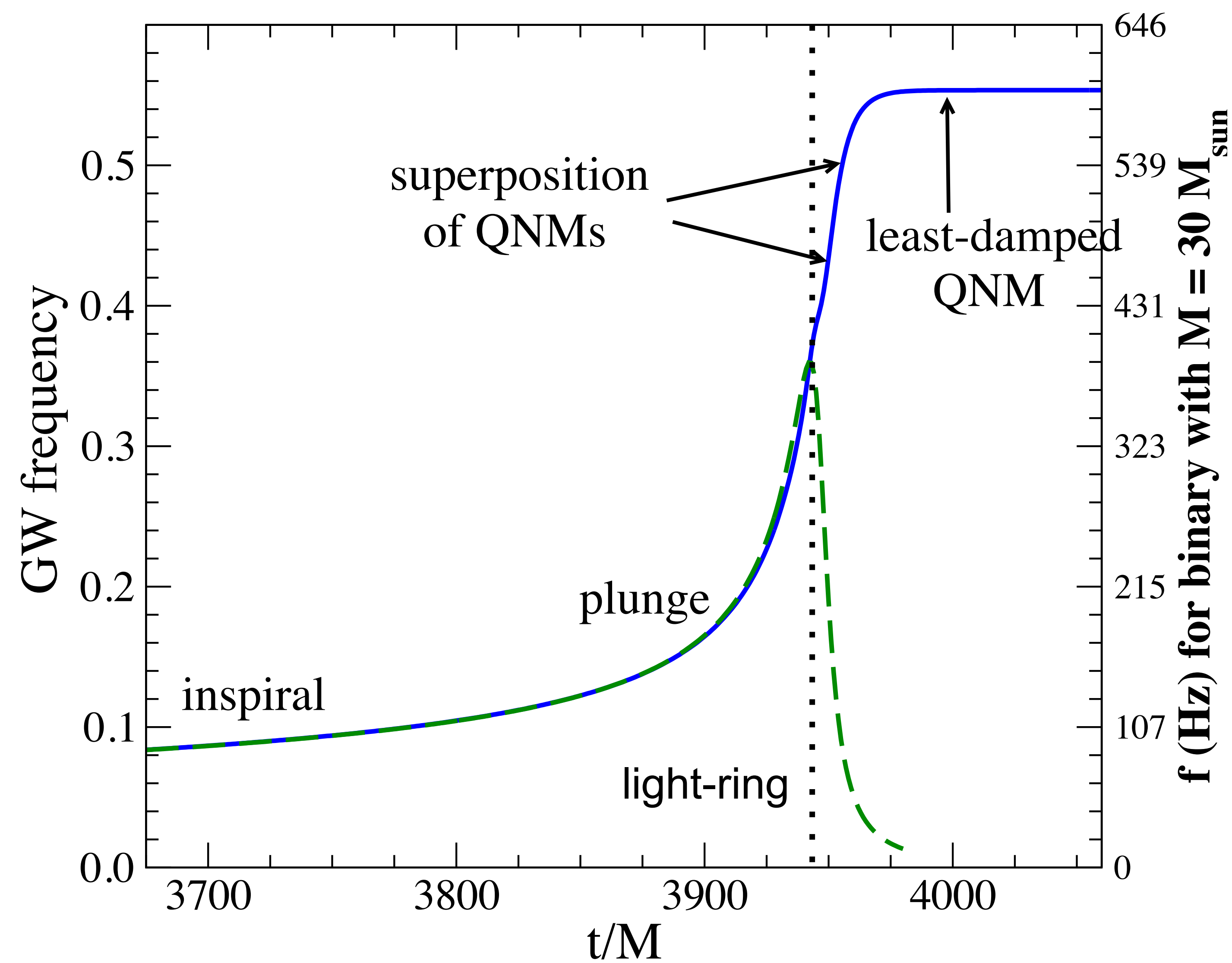
- **Quasi-normal modes** excited at **light-ring crossing**

(Goebel 1972, Davis et al. 1972, Ferrari et al. 1984, Damour et al. 07, Barausse et al. 11, Price et al. 15)

EOB inspiral-merger-ringdown waveform & frequency

… attach **superposition** of **quasi-normal modes** of **remnant** black hole.



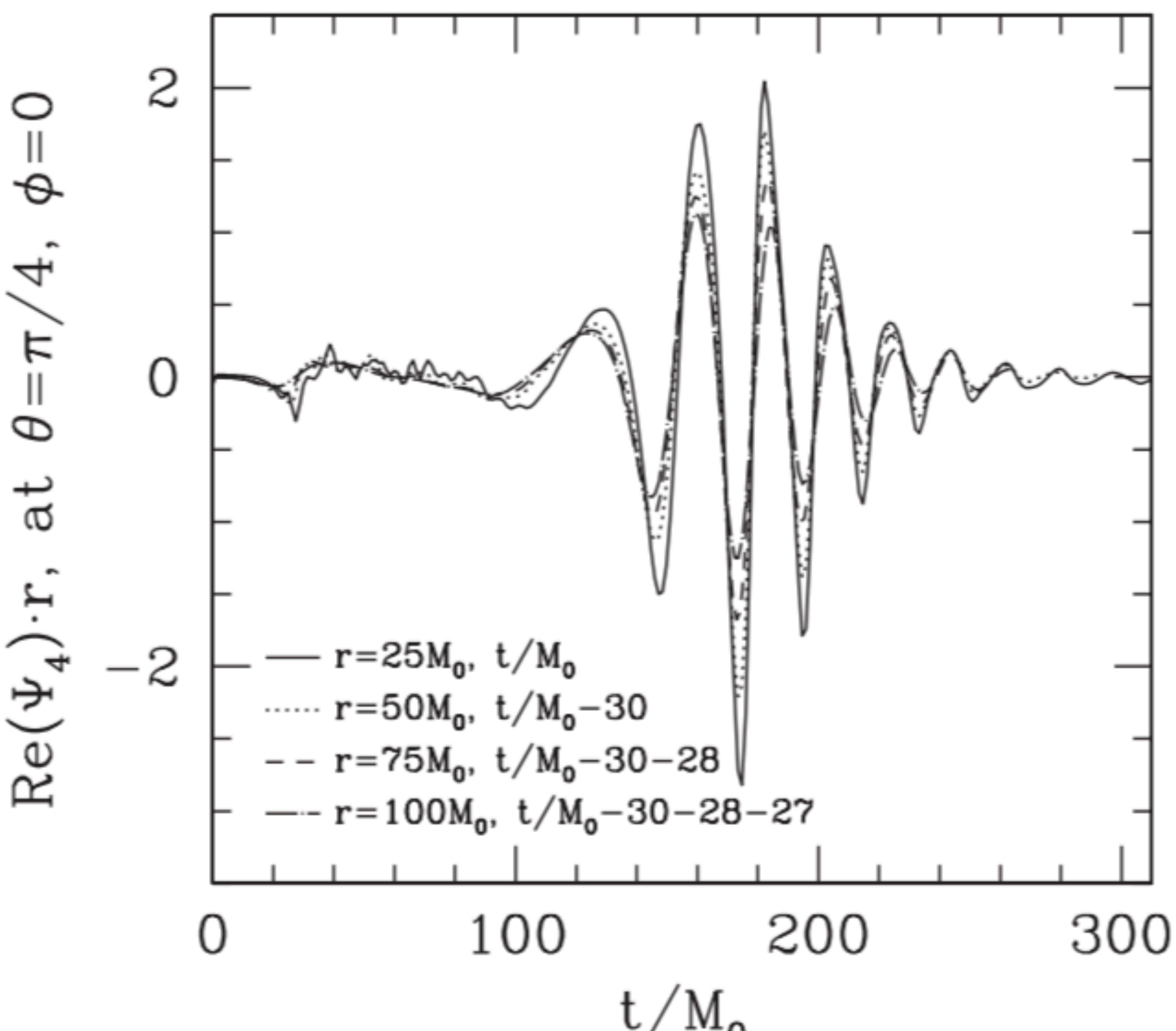


- Very **short/simple transition** plunge-merger-ringdown: “easy” to model!
- Energy **quickly released** during merger: 2%-12%M

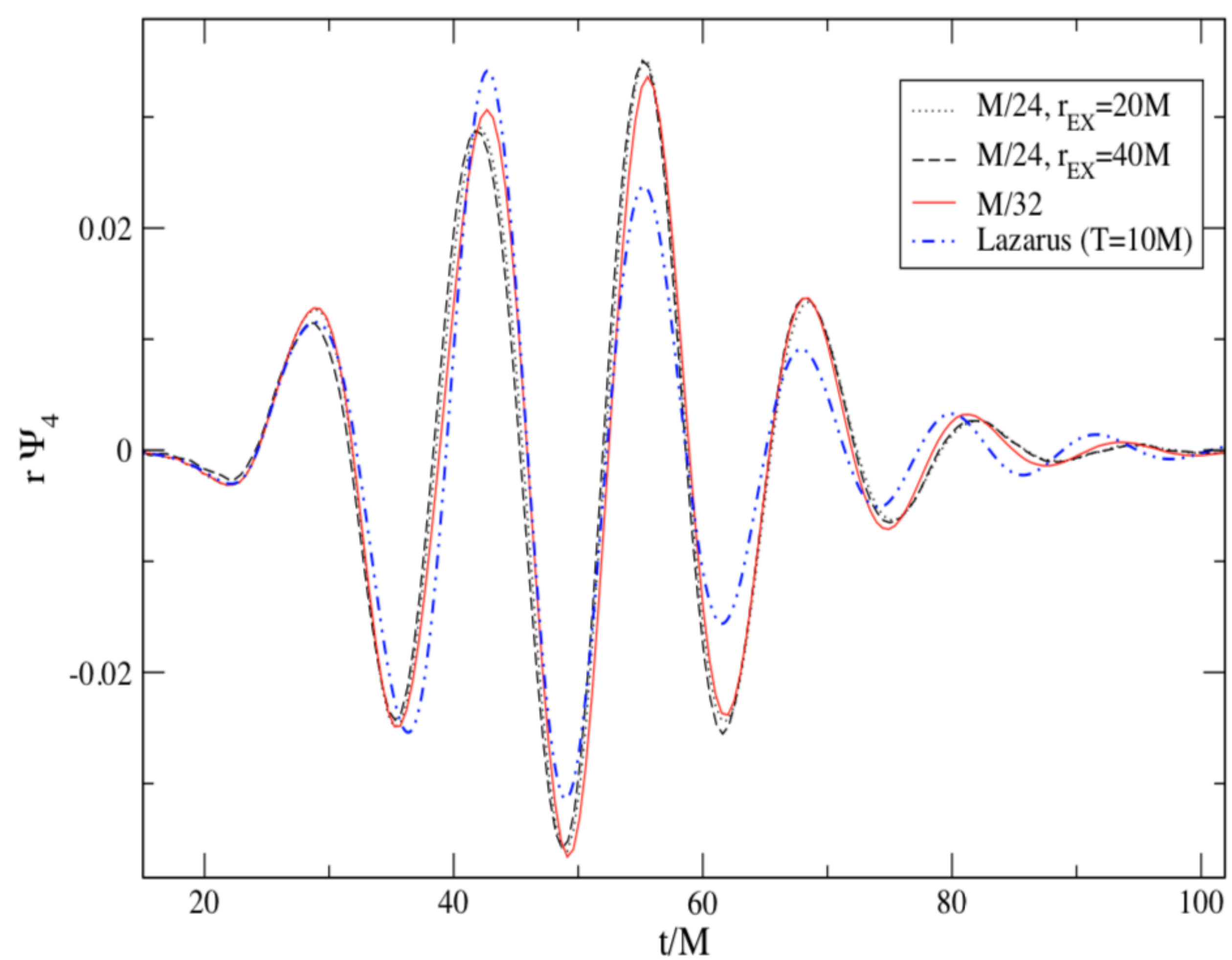
Numerical Relativity: first binary black-hole simulations

- After more than **40 years** of **steady work** (*Choquet-Bruhat, York, Smarr, Price, Pullin, Brügmann, Gundlach, Alcubierre, Brandt, Seidel, Shibata, Nakamura, Baumgarte, Shapiro, Teukolsky,…*) …
- **Breakthrough** in 2005 *(Pretorius 05, Campanelli et al. 06, Baker et al. 06)*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$



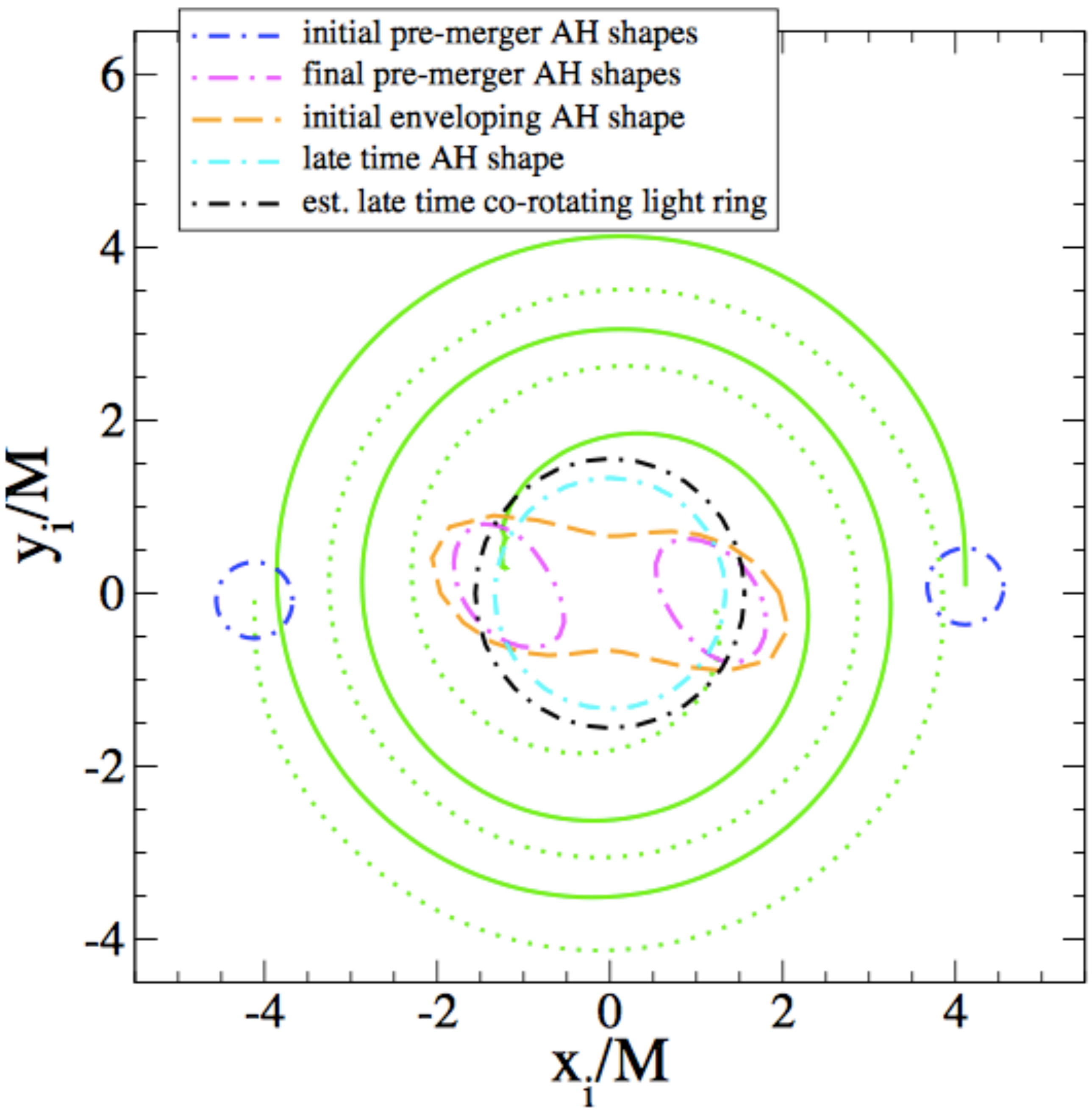
(Pretorius 05)

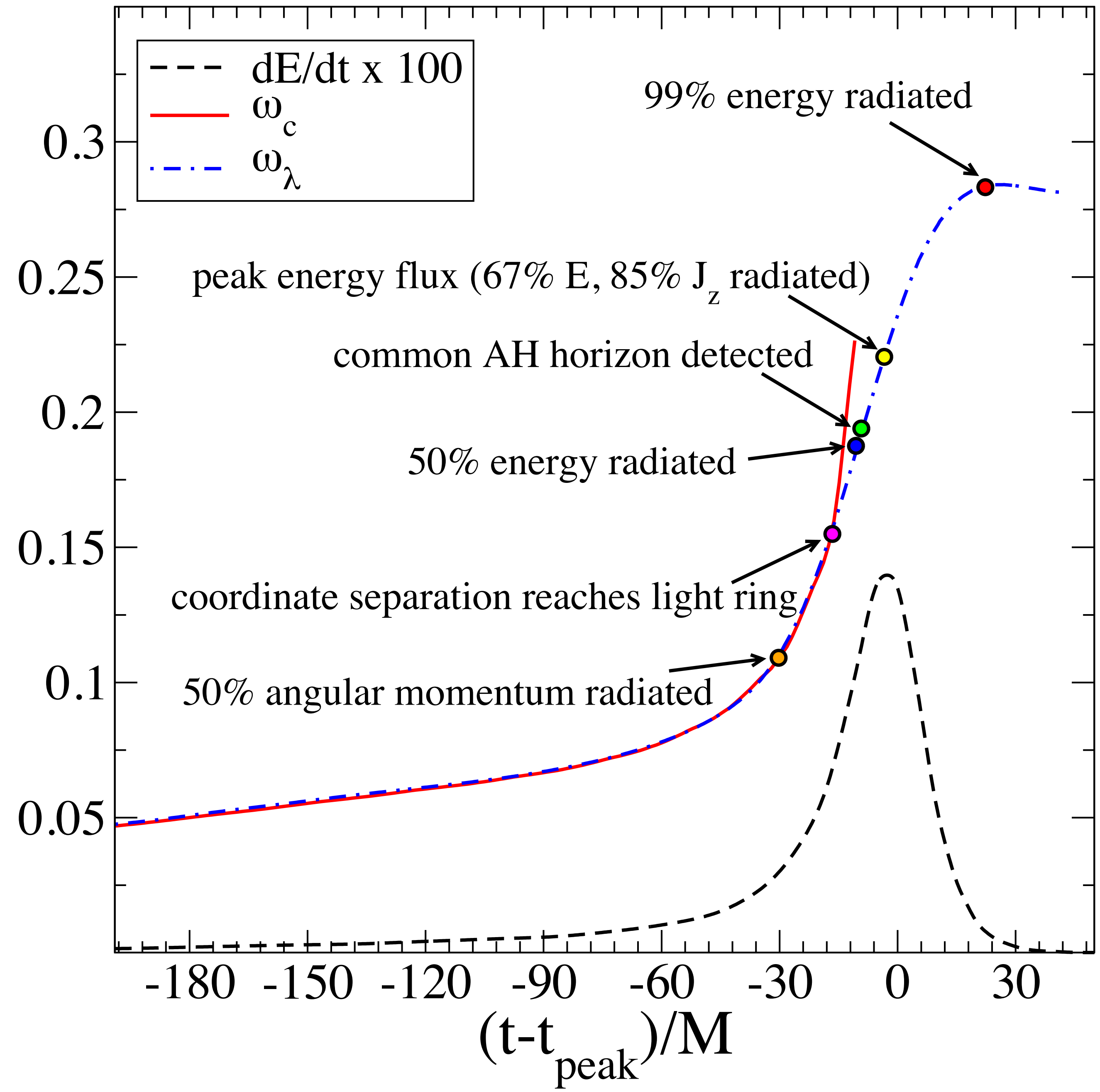


(Baker et al. 06)

The (plunge and) merger in first NR simulations

(AB, Cook & Pretorius 07)



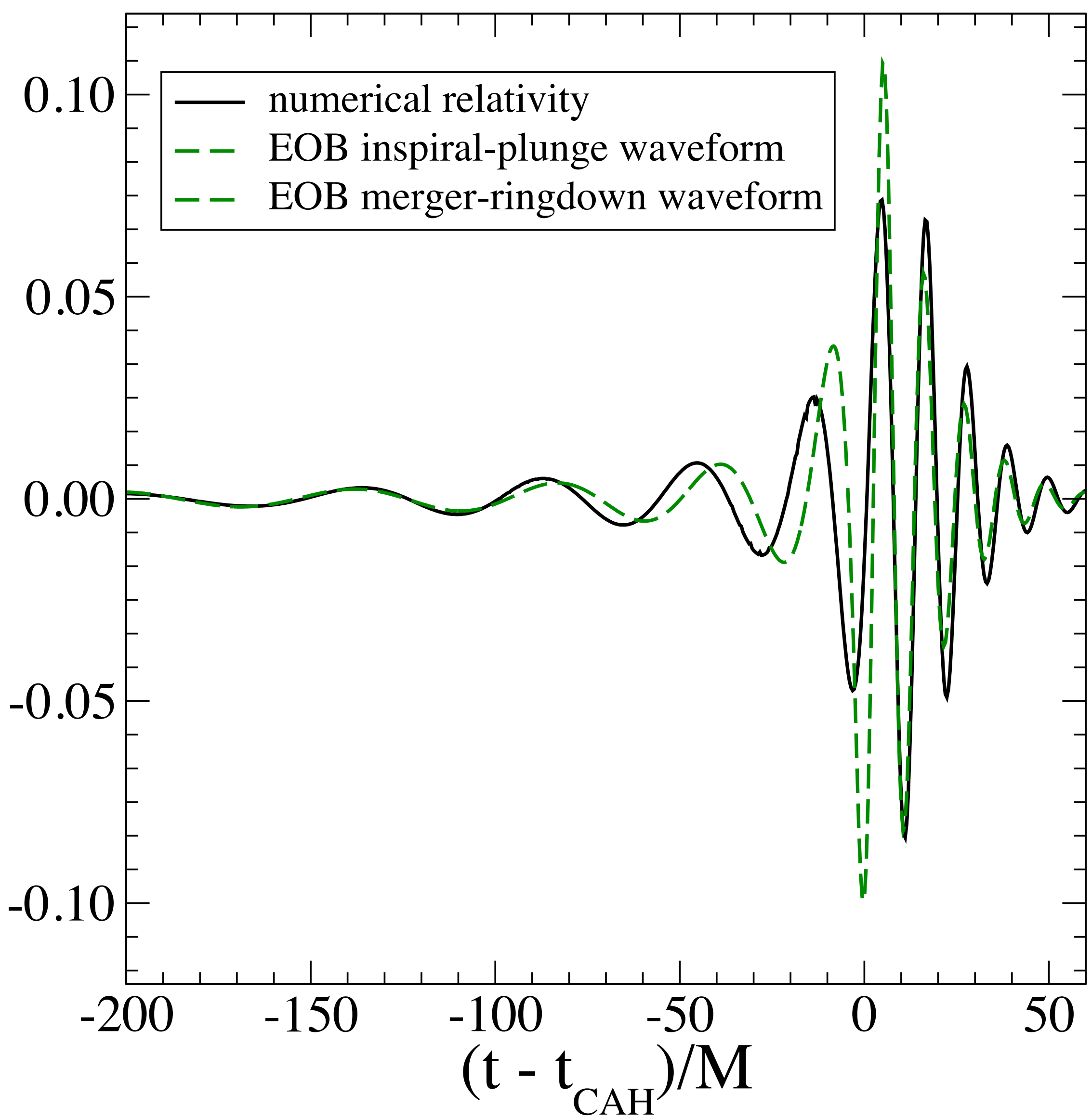


- Very **short/simple transition** plunge-merger-ringdown

- Energy **quickly released** during merger: 2%-12%M

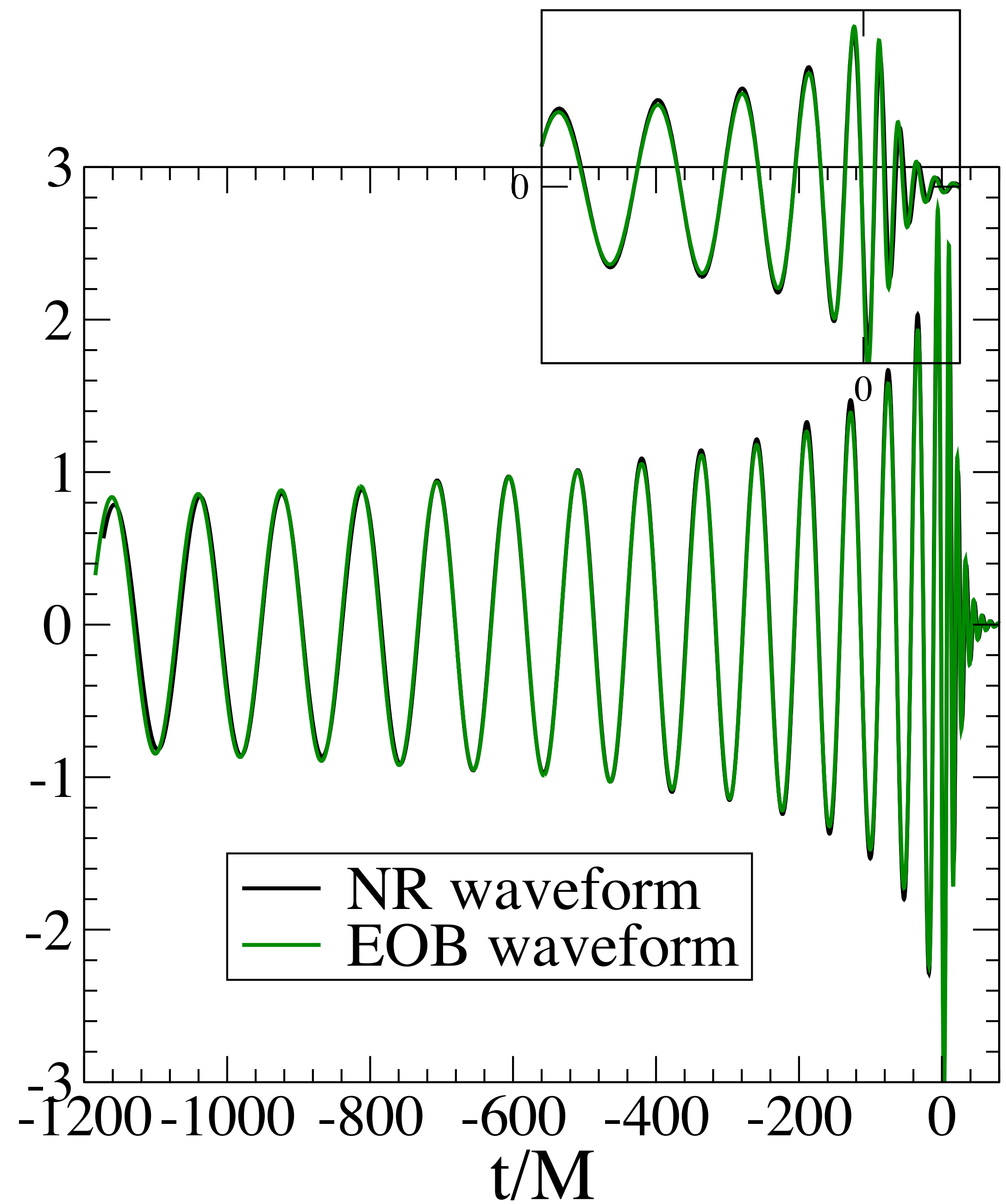
First comparison/calibration between NR and EOB model

- **Uncalibrated** EOB waveform at 3.5PN order



(AB, Cook & Pretorius 07)

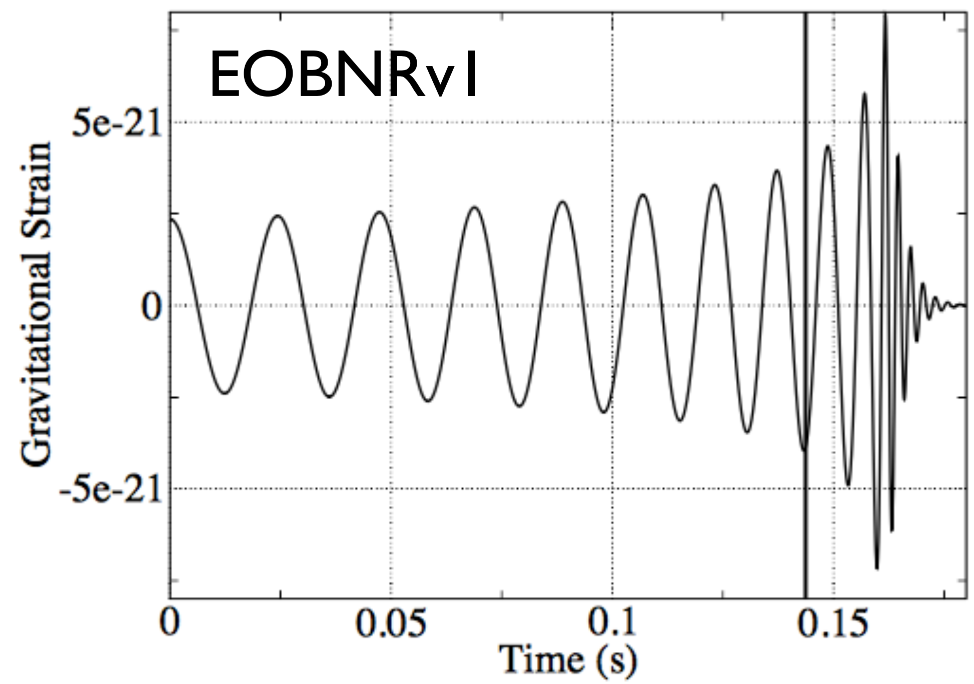
- **Calibrated** EOBNR waveform



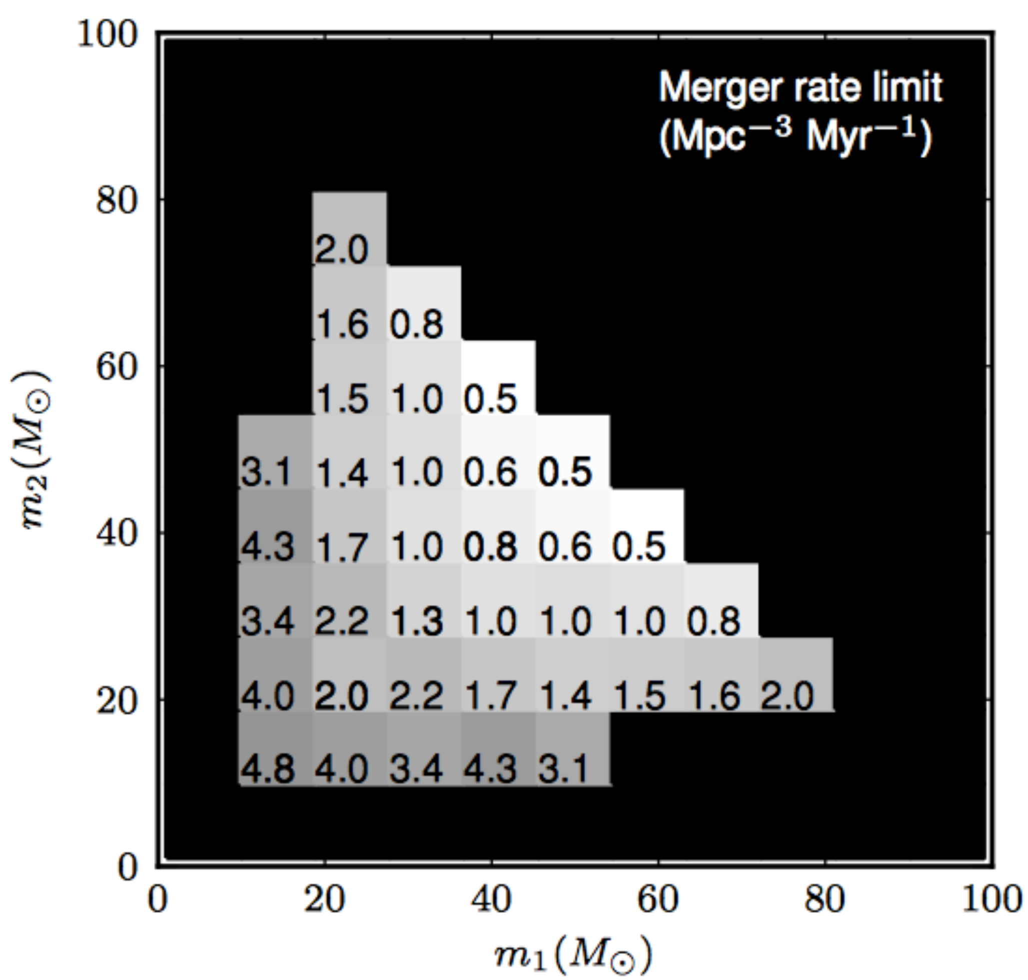
(AB, Pan, Baker, Centrella, Kelly at al. 08)
(see also Damour et al. 08)

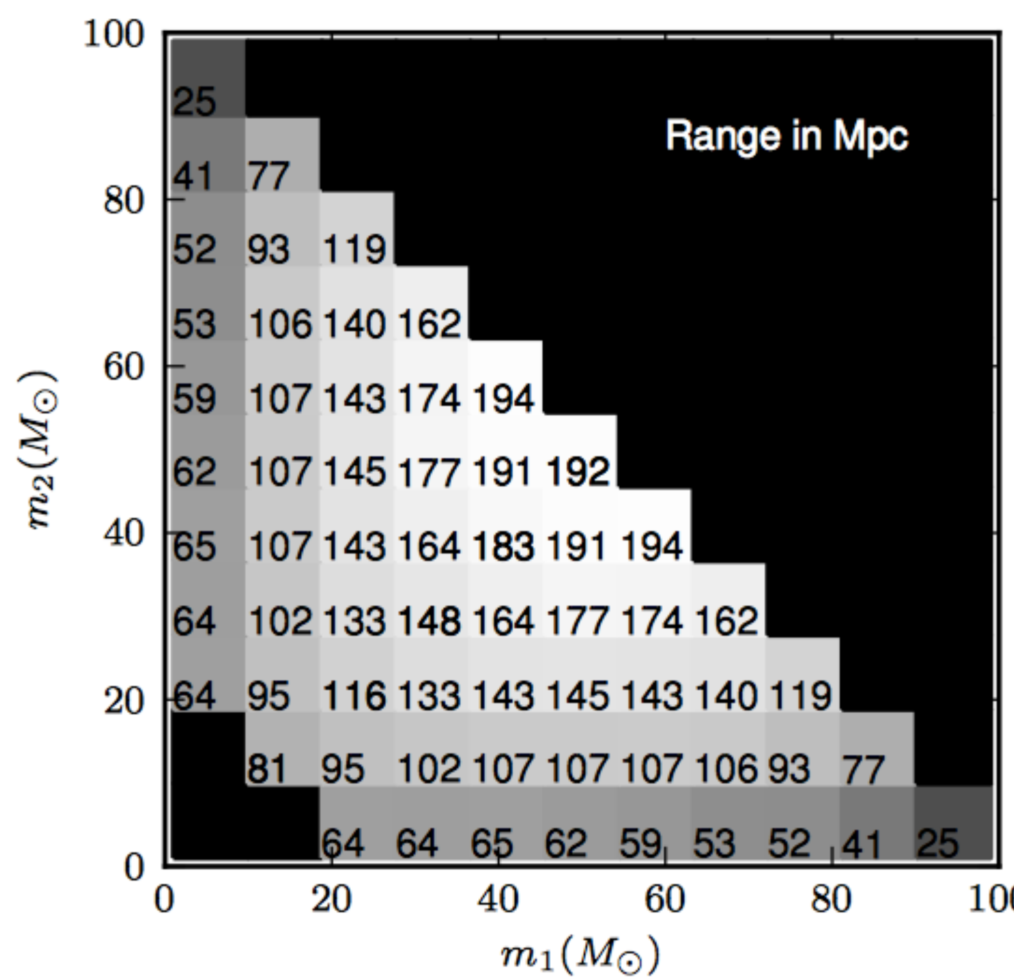
BBH searches in initial LIGO (S5 & S6 runs)

(Abadie et al. PRD83 (2011) 122005, Abadie et al. PRD87 (2013) 022002)



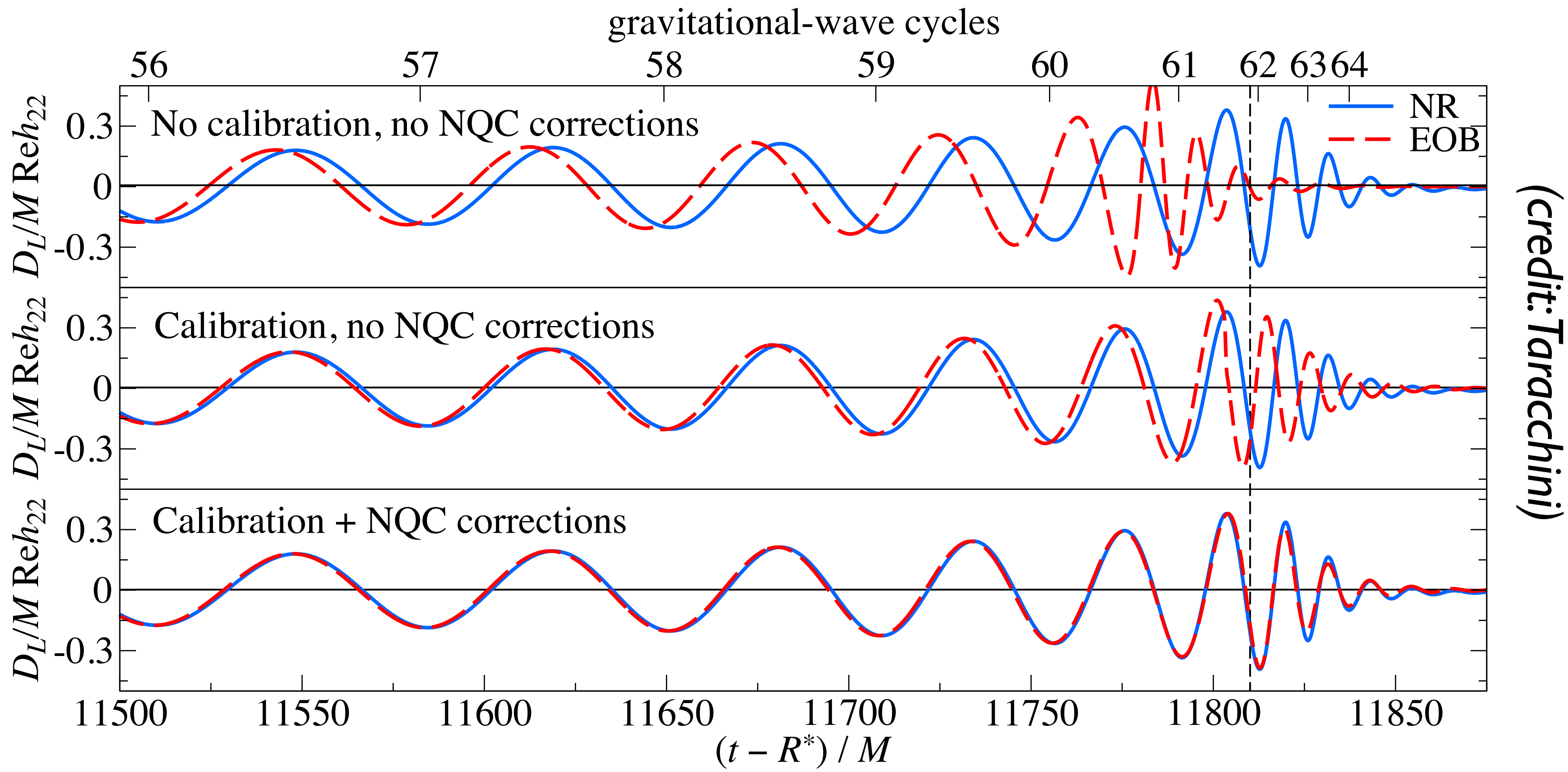
- First **upper limits** from iLIGO



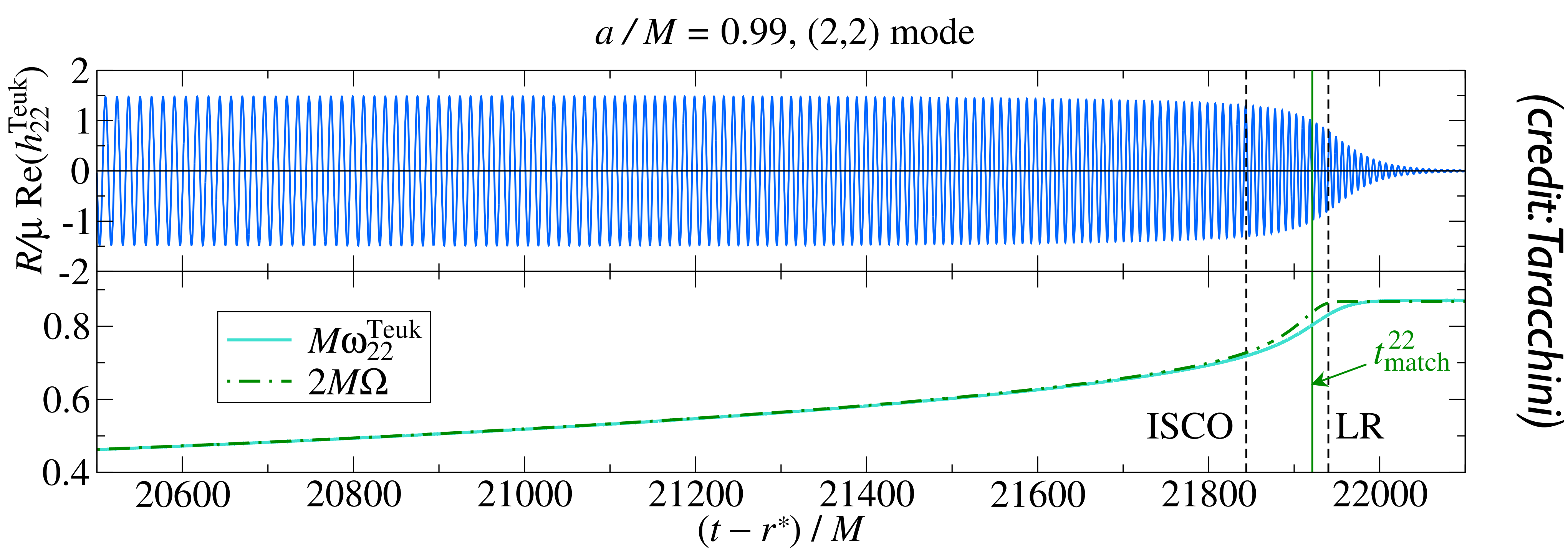


Completing EOB waveforms using NR/perturbation theory information

- We calibrate to inspiral-merger-ringdown **NR** waveforms.



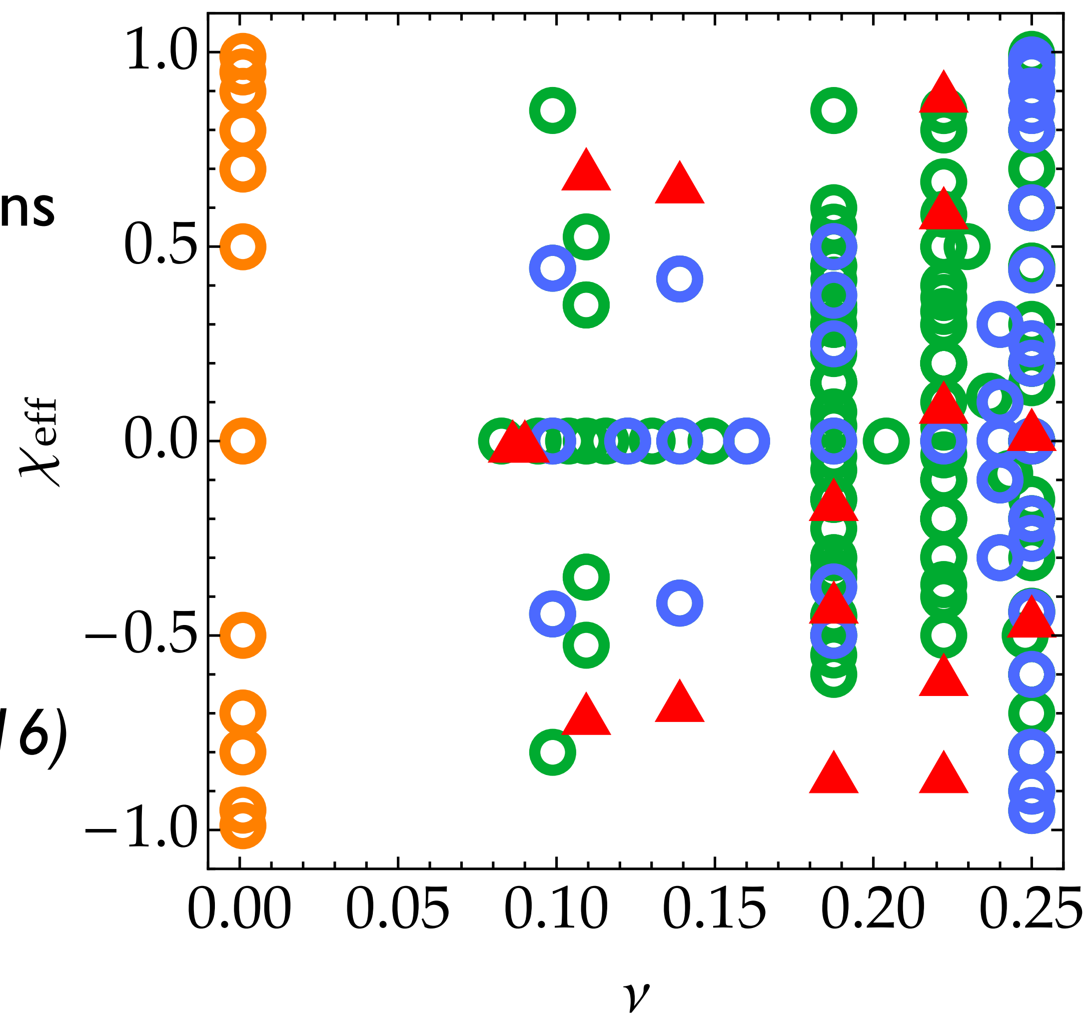
- We calibrate to merger-ringdown **Teukolsky-equation based** waveforms.



Calibration of EOBNR for O1-O3 searches/follow-up analyses

141 SXS simulations

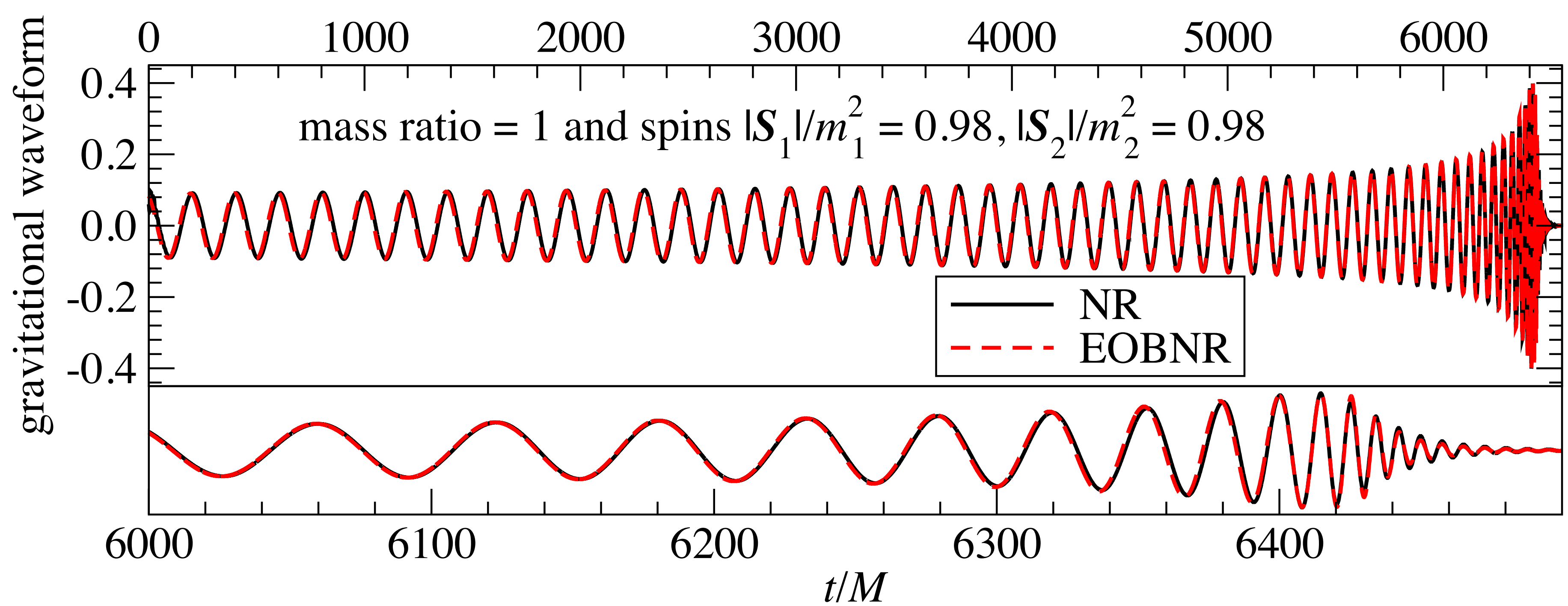
(Bohe' et al. 16)



(Pan, AB et al. 13, Taracchini, AB, Pan, Hinderer & SXS 14, Pürrer 15)

(Bohe', Shao, Taracchini, AB & SXS 16, Babak et al. 16)

(see also Nagar et al. 18)



(Taracchini et al. 14)

Inspiral-merger-ringdown phenomenological waveforms

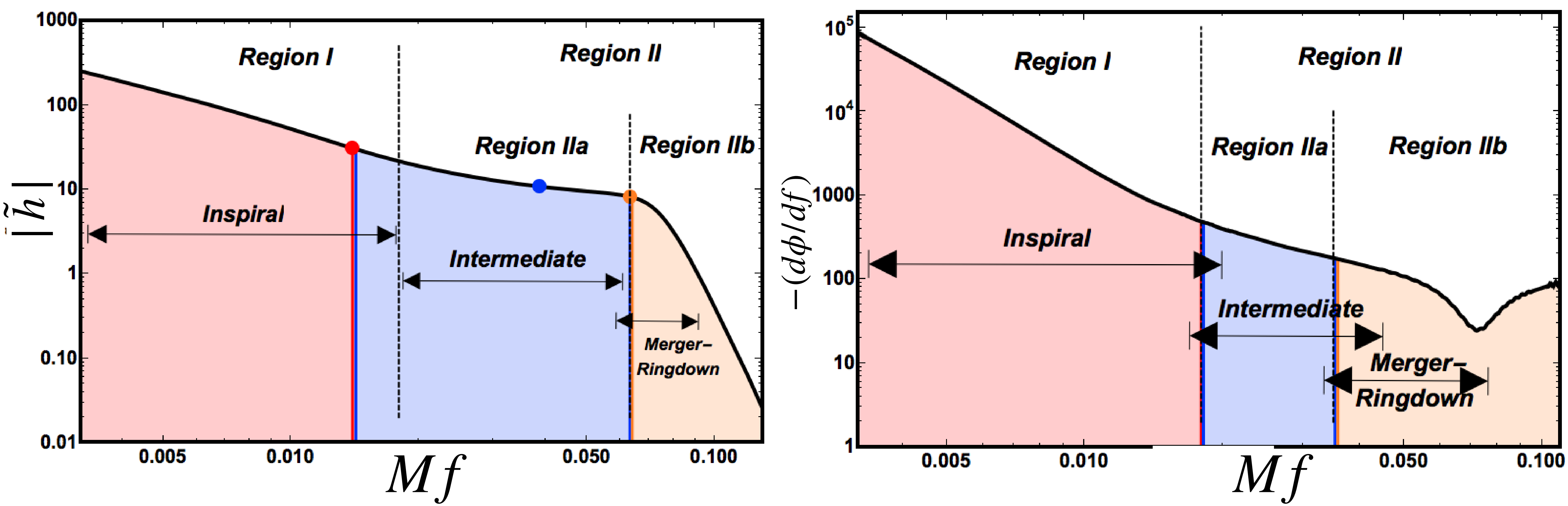
Phenomenological waveforms used in O1-O3 follow-up analyses

- First works in mid-late 2000 *(Ajith et al. 07, Pan et al. 07, Santamaria, Ohme et al. 10)*

(If PN were used instead, accuracy will degrade, because of "gap" between PN and NR)

- **Fast, frequency-domain** waveform model hybridizing EOB & NR waveforms, and then fitting *(Schmidt et al.12; Hannam et al. 13; Khan et al. 15; Husa et al. 15; Khan et al. 18-19; García-Quíros et al. 20, Pratten et al. 20)*

$$\tilde{h}(f;\lambda_i) = \mathcal{A}(f;\lambda_i)\, e^{i\phi(f;\lambda_i)}$$ (IMRPhenom)



On phenomenological inspiral-merger-ringdown waveforms

(Santamaria, Ohme et al. 10)

$$\tilde{h}(f;\lambda_i) = \mathcal{A}(f;\lambda_i)\, e^{i\phi(f;\lambda_i)}$$

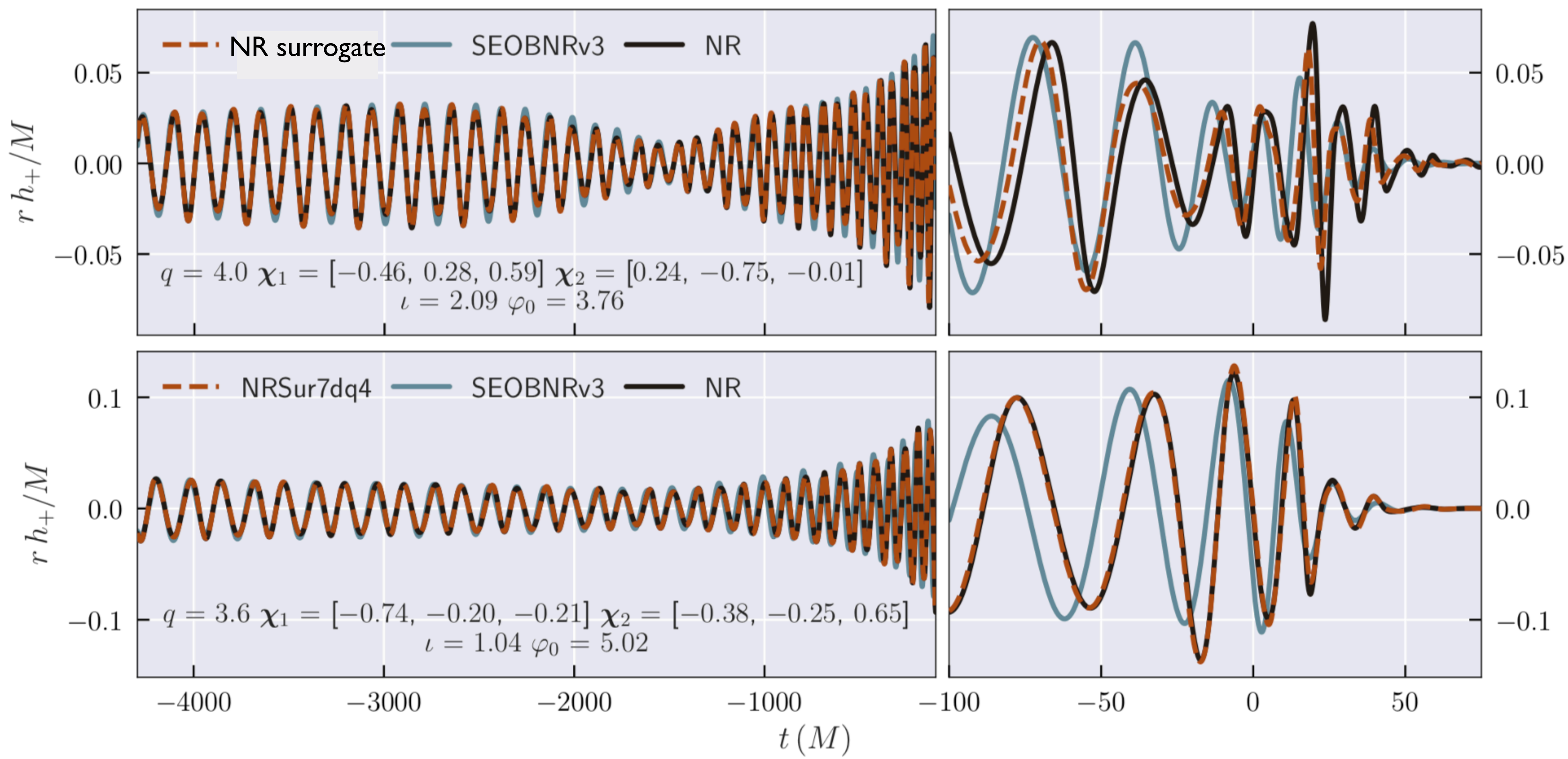
$$\mathcal{A}(f;\lambda_i) \equiv C \begin{cases} \left(\frac{\pi M f}{a_0\nu^2+b_0\nu+c_0}\right)^{-7/6} & \text{if } f < \frac{a_0\nu^2+b_0\nu+c_0}{\pi M} \\ \left(\frac{\pi M f}{a_0\nu^2+b_0\nu+c_0}\right)^{-2/3} & \text{if } \frac{a_0\nu^2+b_0\nu+c_0}{\pi M} \le f < \frac{a_1\nu^2+b_1\nu+c_1}{\pi M} \\ w\,\mathcal{L}\left(f, \frac{a_1\nu^2+b_1\nu+c_1}{\pi M}, \frac{a_2\nu^2+b_2\nu+c_2}{\pi M}\right) & \text{if } \frac{a_1\nu^2+b_1\nu+c_1}{\pi M} \le f < \frac{a_3\nu^2+b_3\nu+c_3}{\pi M} \end{cases}$$

$$\phi(f;\lambda_i) = 2\pi f t_0 + \varphi_0 + \frac{1}{\nu}\sum_{k=0}^{7}(x_k\,\nu^2 + y_k\,\nu + z_k)\,(\pi M f)^{(k-5)/3}$$

(Schmidt et al. 12; Hannam et al. 13; Khan et al. 15; Husa et al. 15; Khan et al. 18-19; García-Quíros et al. 20, Pratten et al. 20)

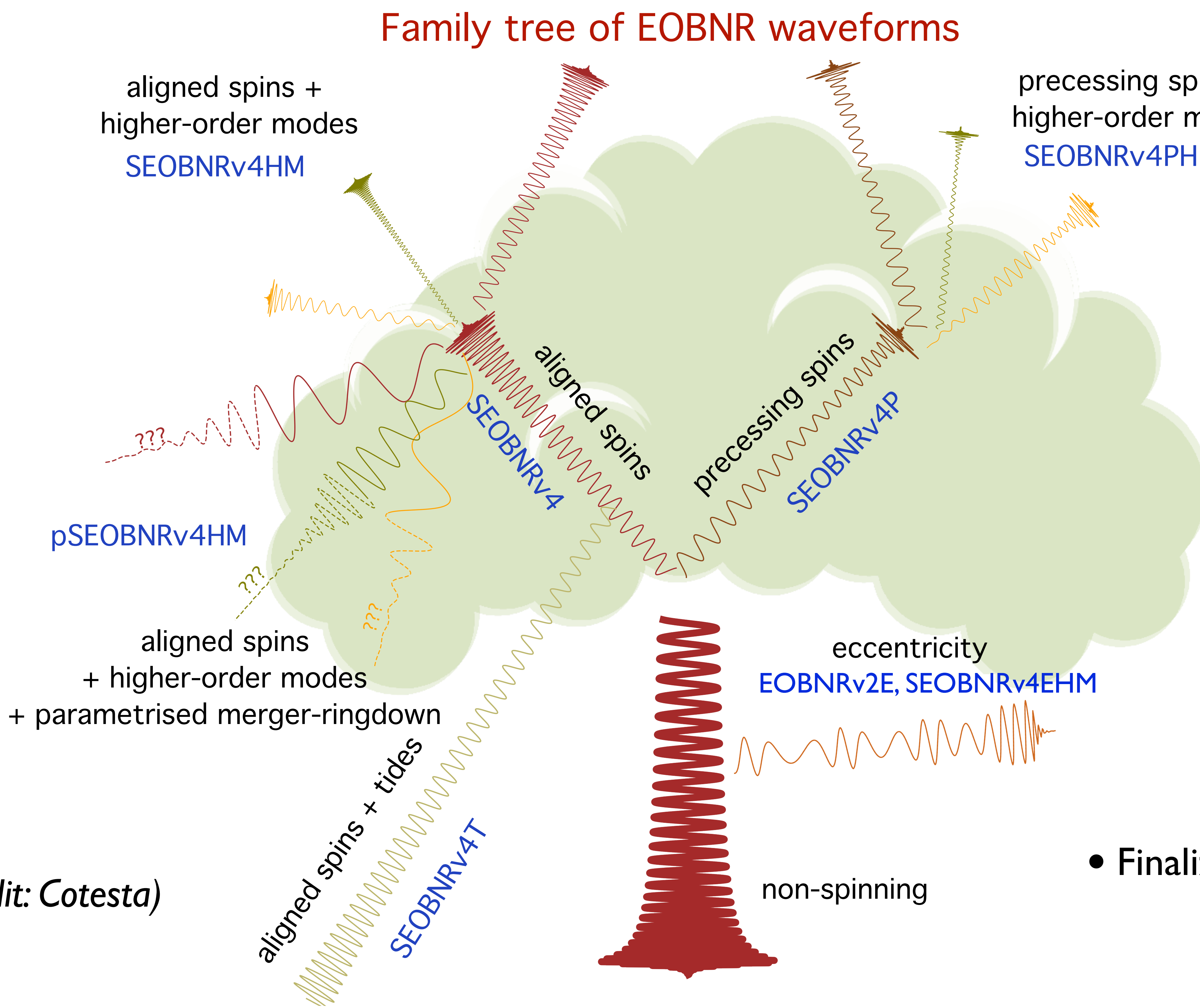
Surrogate models using NR simulations

- First works in late 2000 *(Field, Galley, Tiglio; Blackman, Varma, Scheel, …)*
- **NR surrogate models** are built **directly** by interpolating **NR simulations,** which are **selected in parameter space** using analytical waveform models.
- **Highly accurate**, but **limited** in binary's parameter space and length (~20 orbits).



(Varma et al. 19)

Ever more physics in waveform models



Other family trees for IMRPhenom & TEOBResumS

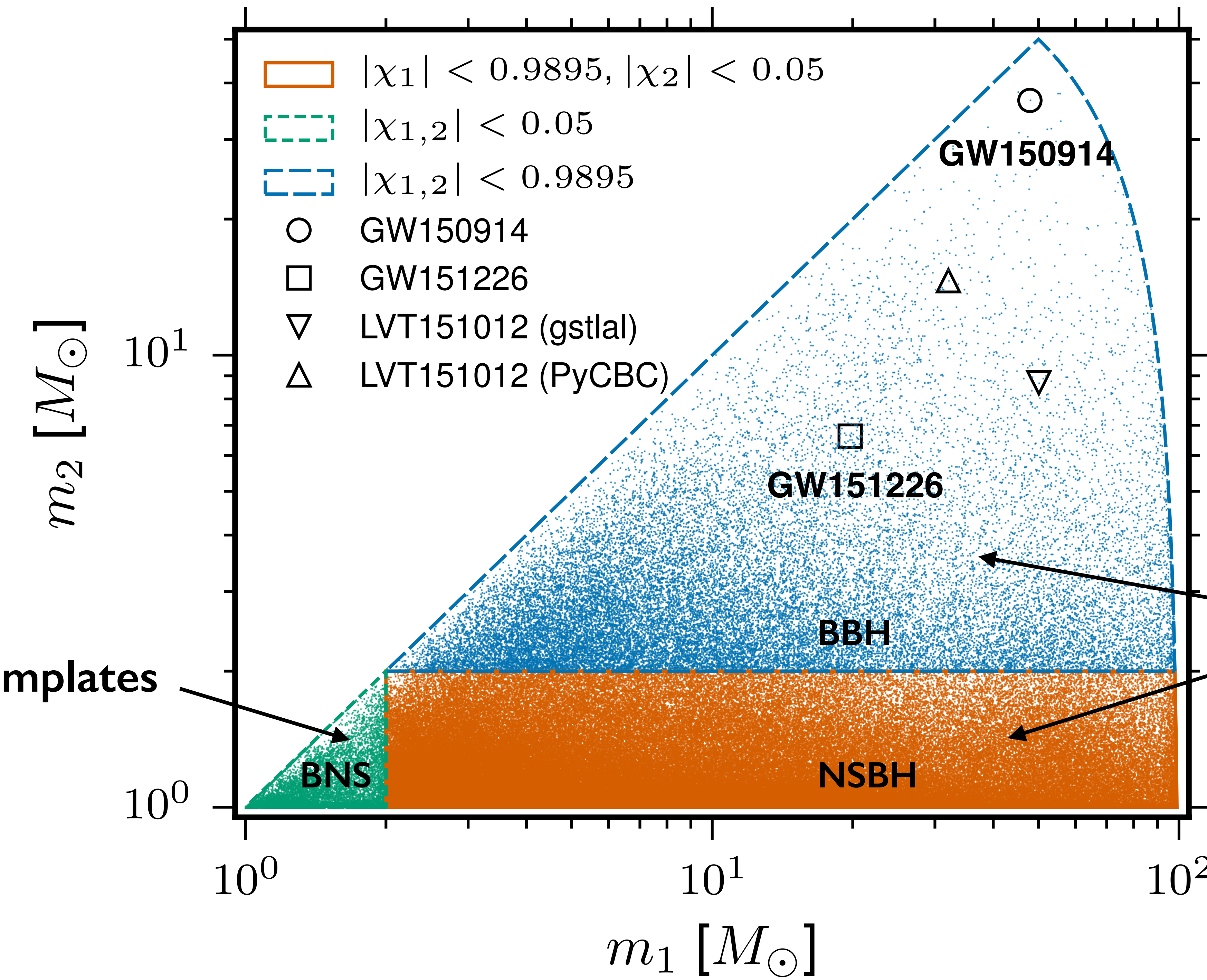
(credit: Cotesta)

- Finalizing **SEOBNRv5!**

Science with GW observations

Template bank used in LIGO first observing run (O1)

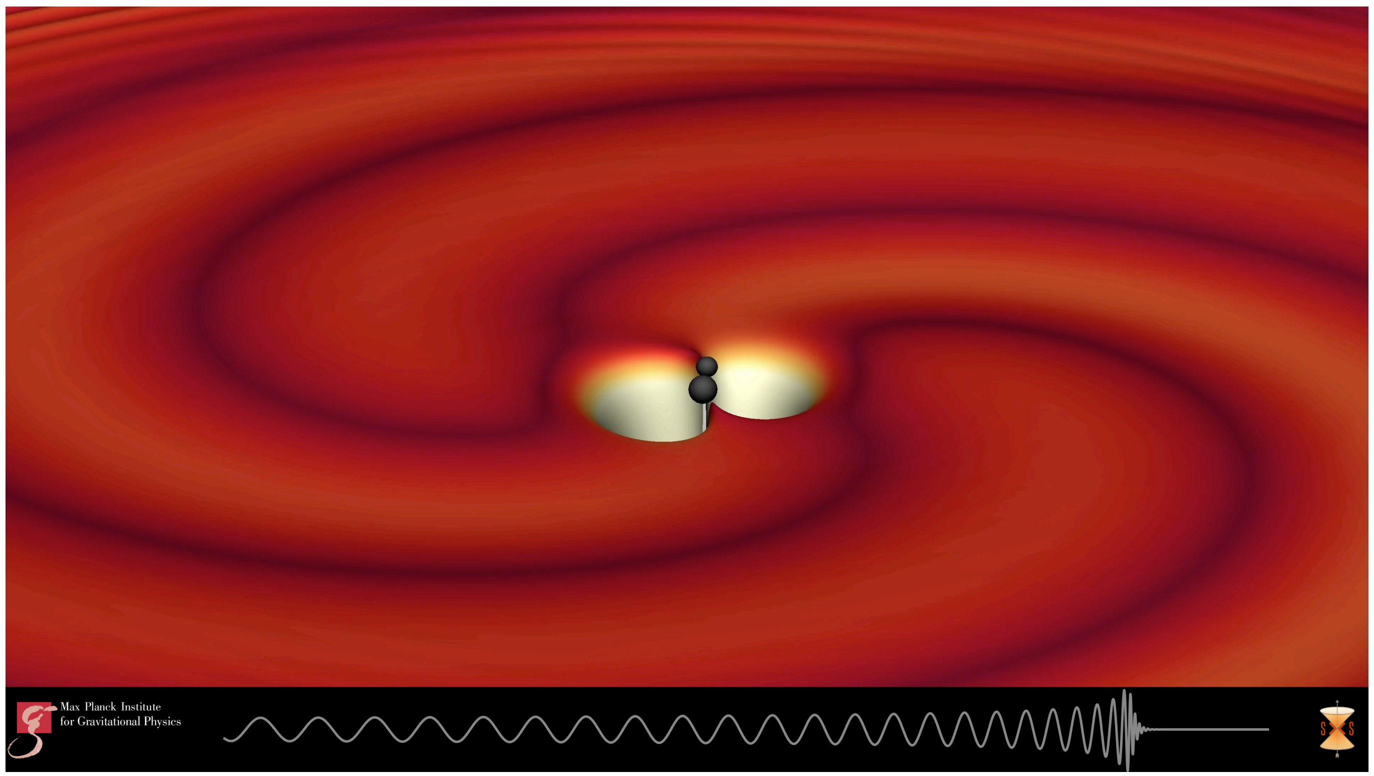
(Abbott et al. Phys.Rev. X6 (2016) no.4, 041015)



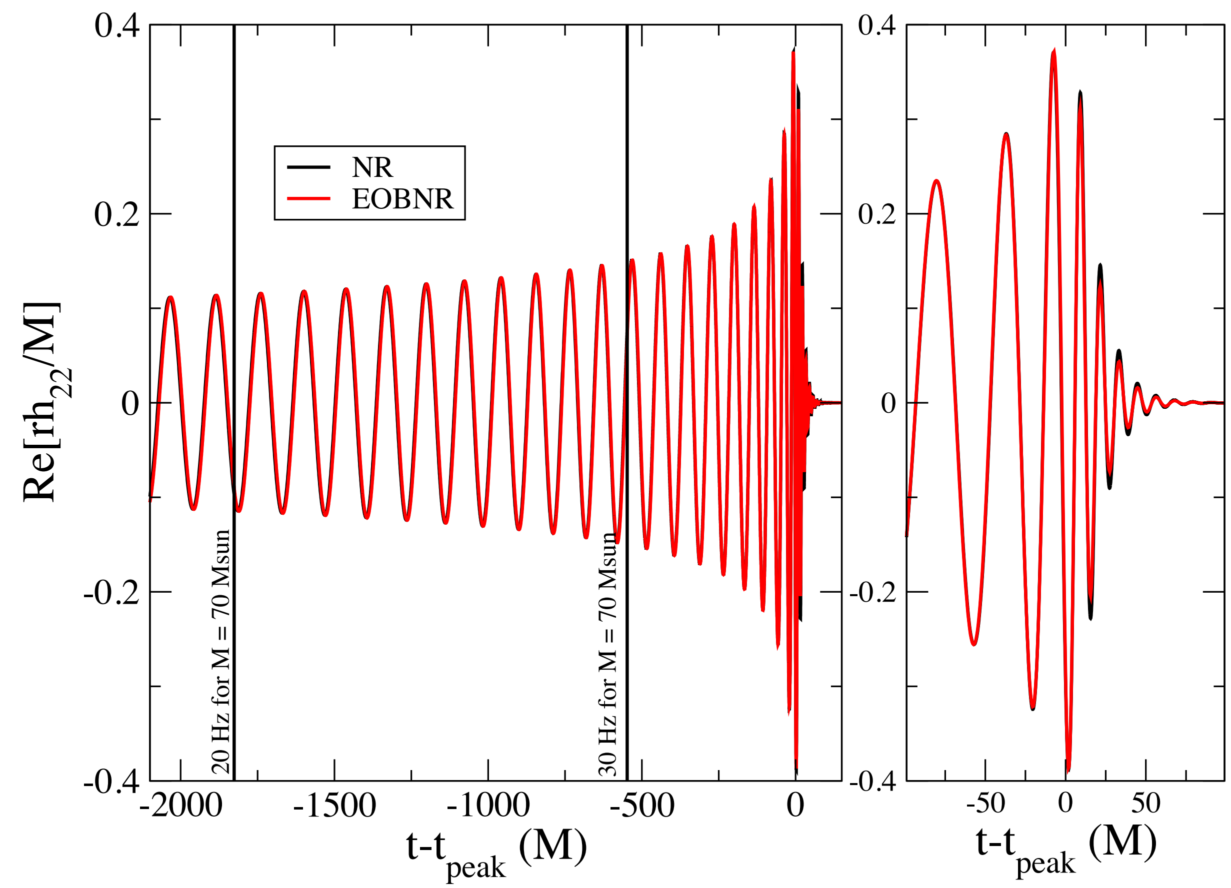
PN SPA templates
~40,000

EOBNR templates
~200,000

NR simulation of GW150914 & comparison with EOBNR



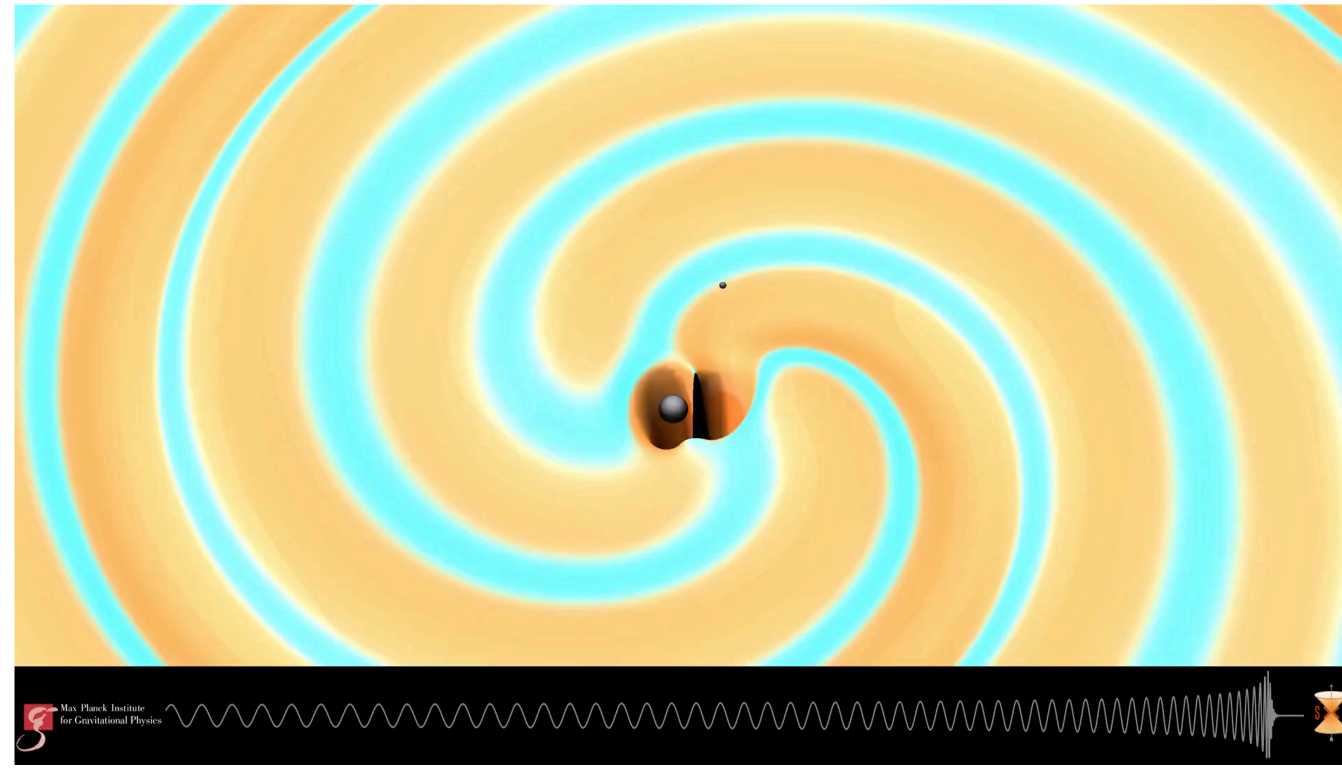
(Ossokine, AB & SXS)



- Waveform models very closely match the exact solution from Einstein equations around GW150914 & GW151226.

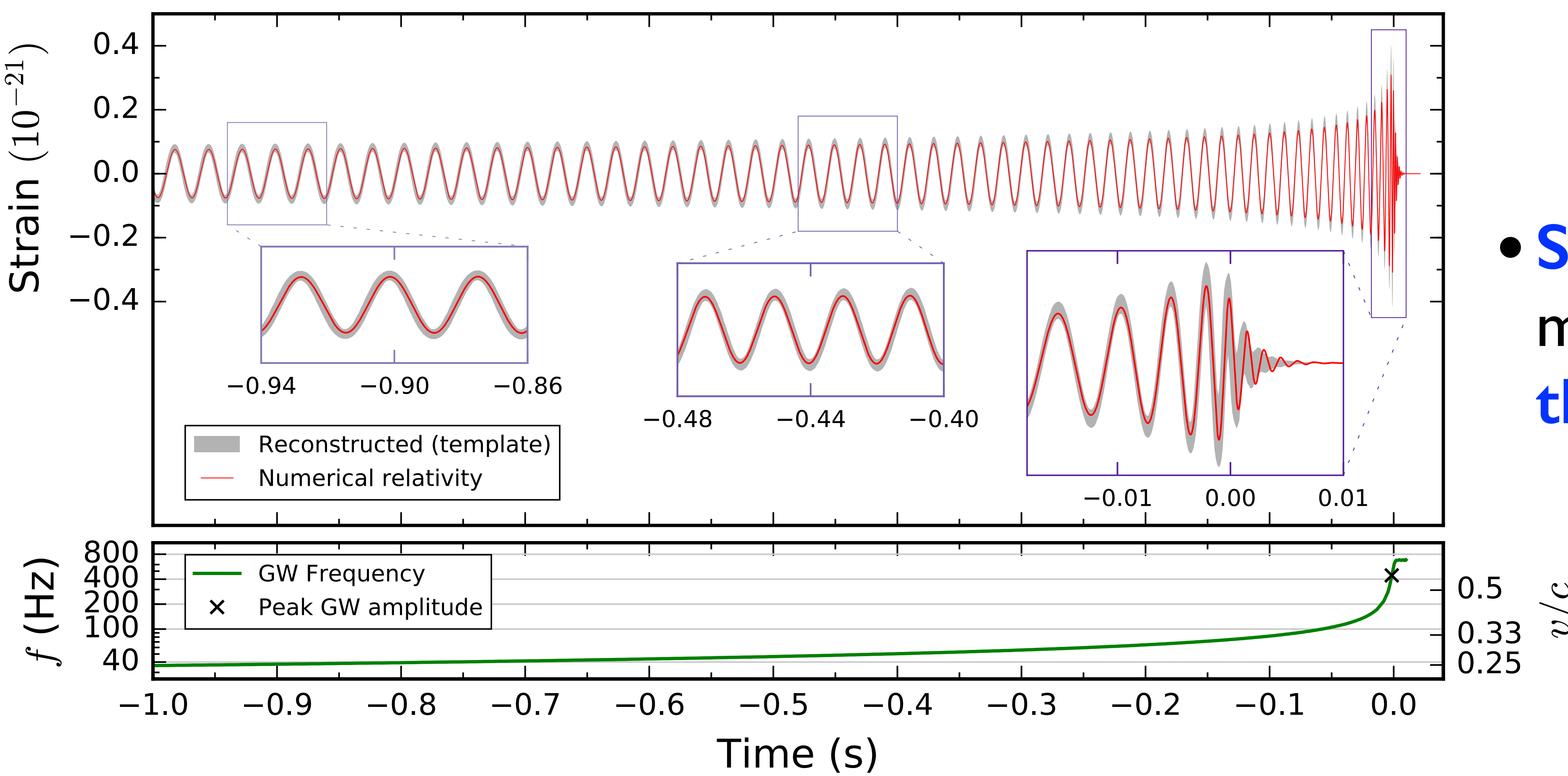
- Systematics due to modeling are smaller than statistical errors.

NR simulation of GW151226 & comparison with EOBNR



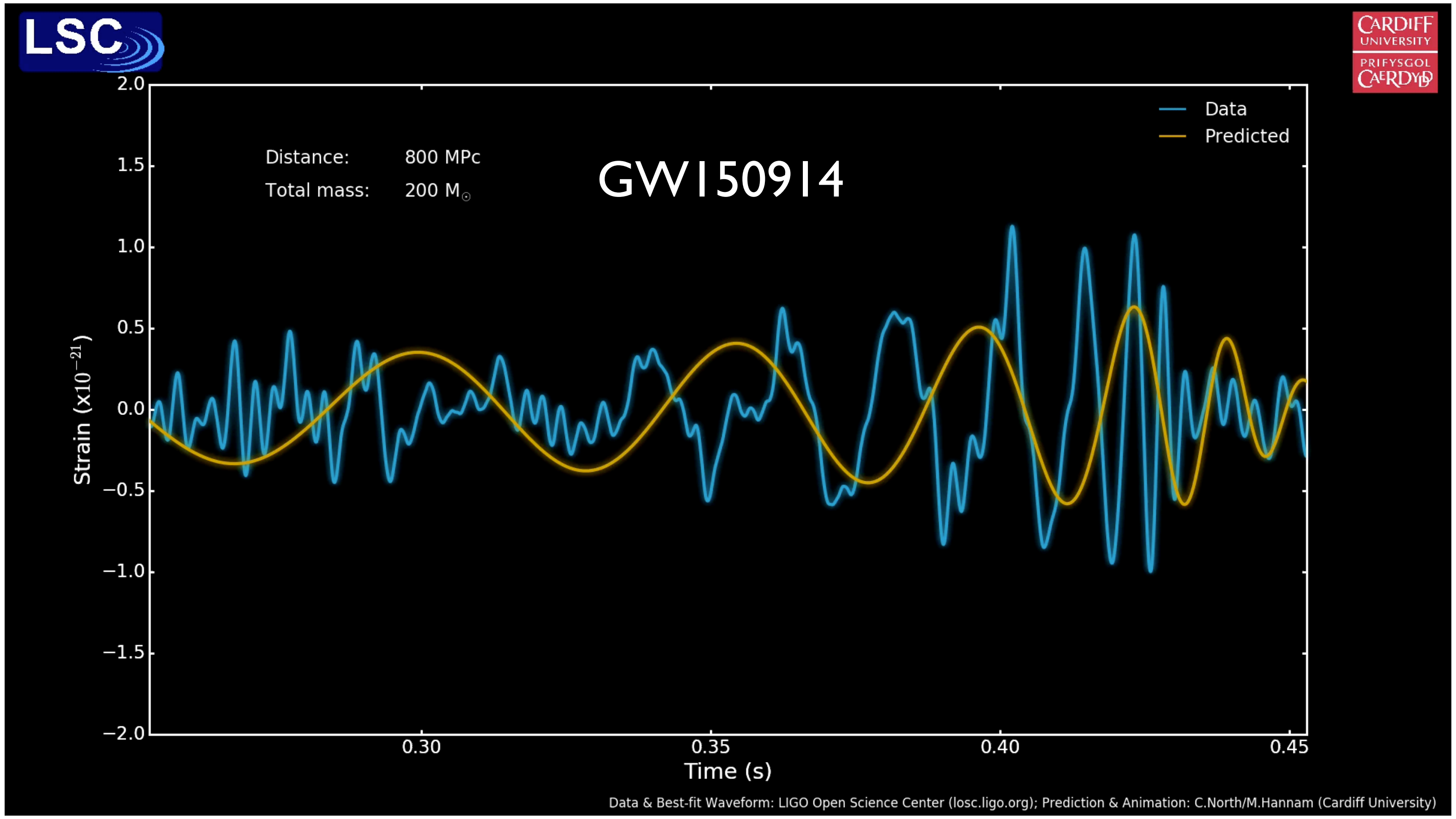
(Ossokine, AB & SXS)

(Abbott et al. PRL 116 (2016) 241103)



- **Systematics** due to modeling **are smaller than statistical** errors.

Matching models against data to unveil source properties



- **GW150914** took place **1.4 billion** light-years away.

- Binary's **component** black holes:
  - m_1 = 36 solar mass
  - m_2 = 29 solar mass

- **Final** black hole:
  - m = 62 solar mass
  - intrinsic rotation= **67% of maximum value**

Inferring sources' properties/model selection upon detection

- **Bayes theorem**:

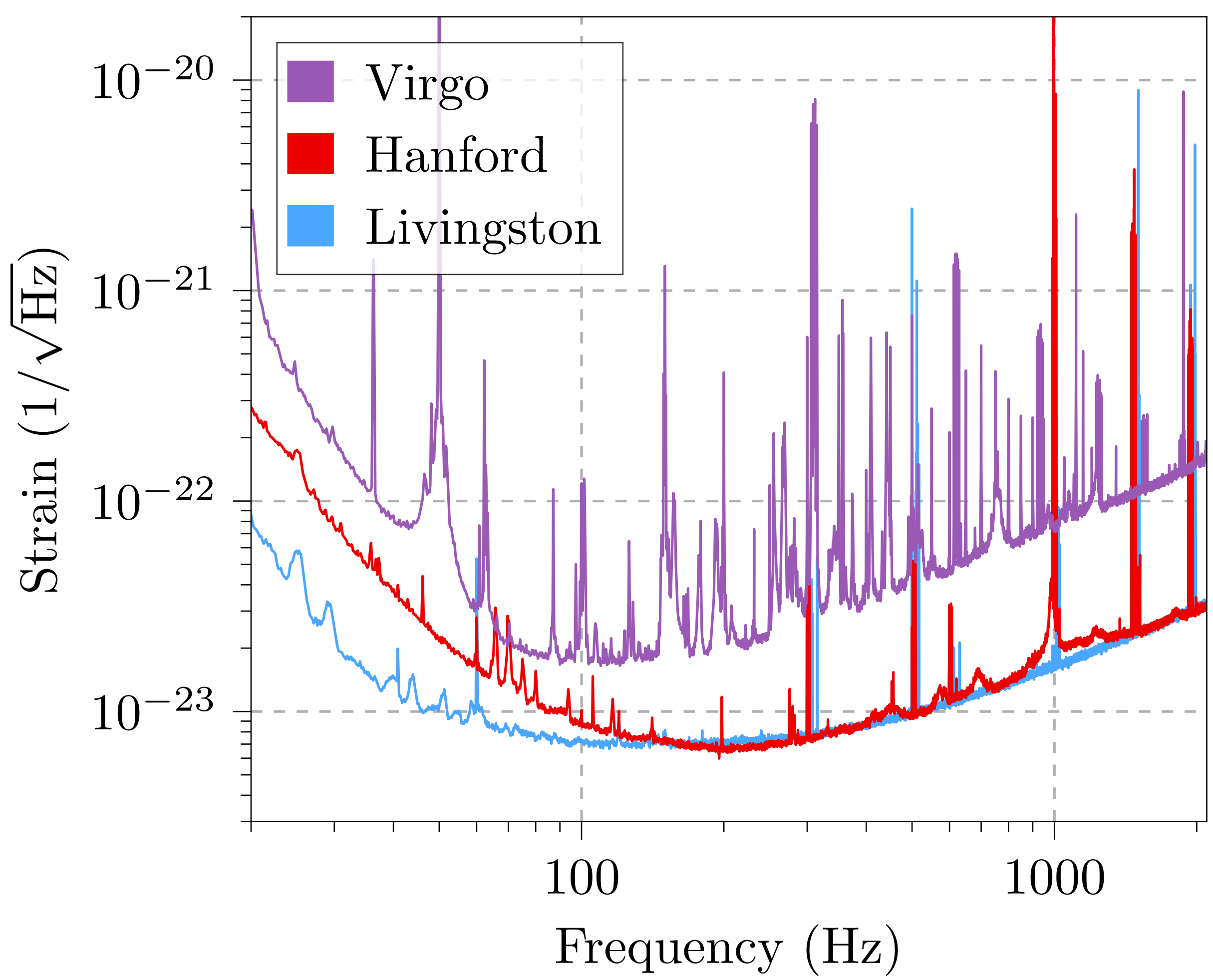
Likelihood function ↓ ↓ prior probability

$$\mathcal{P}(\theta|d,\mathcal{H}) \propto \Lambda(d|\theta,\mathcal{H}) \times \mathcal{P}(\theta,\mathcal{H})$$

↑
posterior probability distribution

- **Likelihood function** for observed data d(t) = n(t) + h(t), given hypothesis that there is GW signal with parameters θ:

$$\Lambda(d|\theta,\mathcal{H}) \propto \exp\left(-2\sum_i \frac{|\tilde{d}(f_i) - \tilde{h}(f_i;\theta)|^2}{S_n(f_i)}\right)$$



(Abbott et al. PRL 119 (2017) 141101)

Inferring sources' properties/model selection upon detection

- **Bayes theorem**:

Likelihood function — prior probability

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posterior probability distribution

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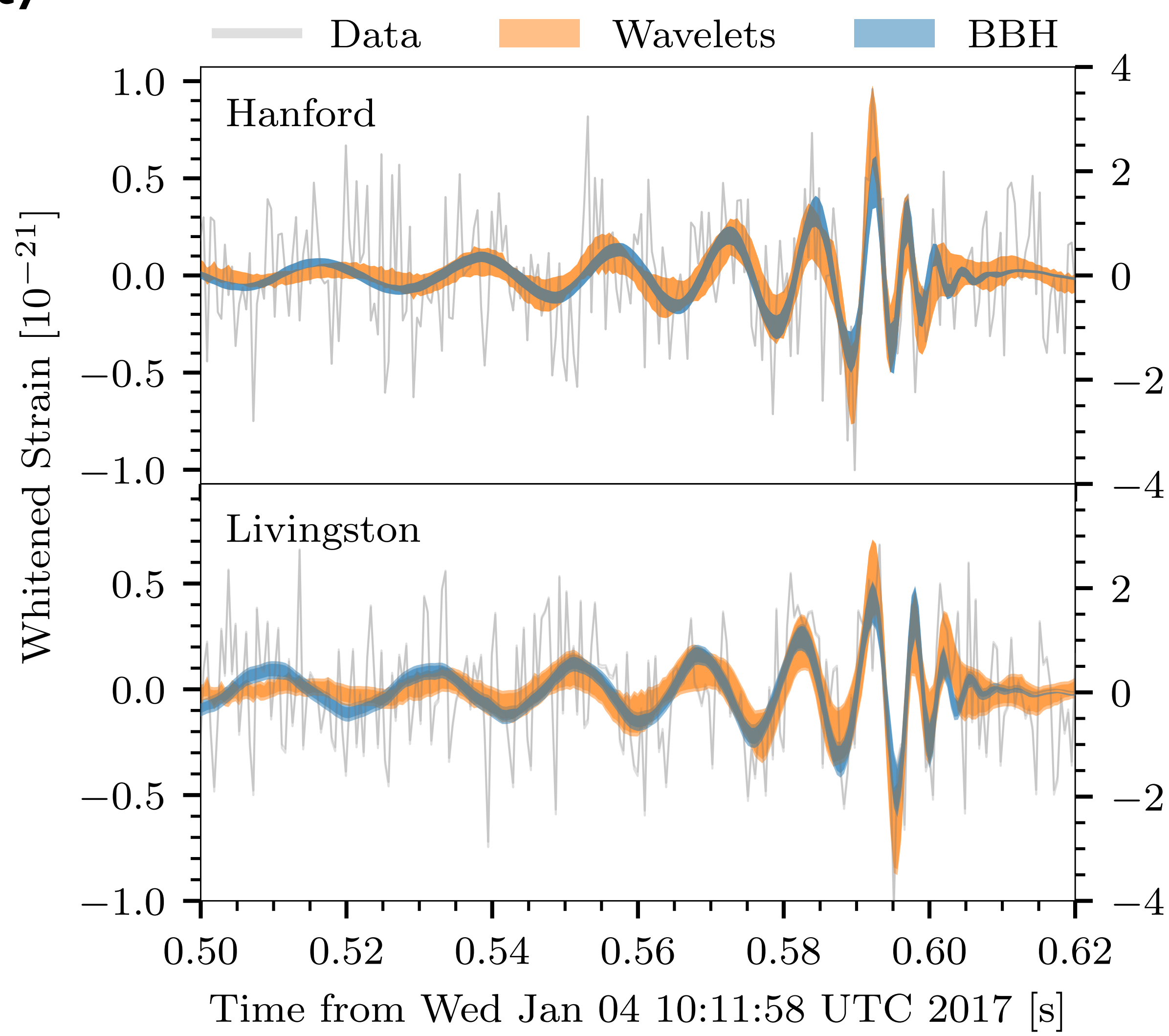
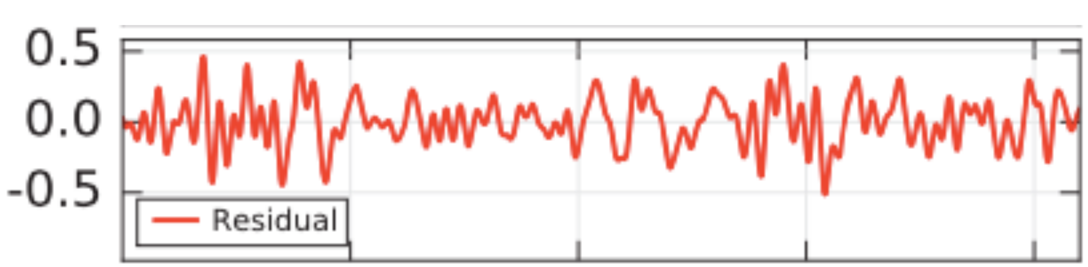
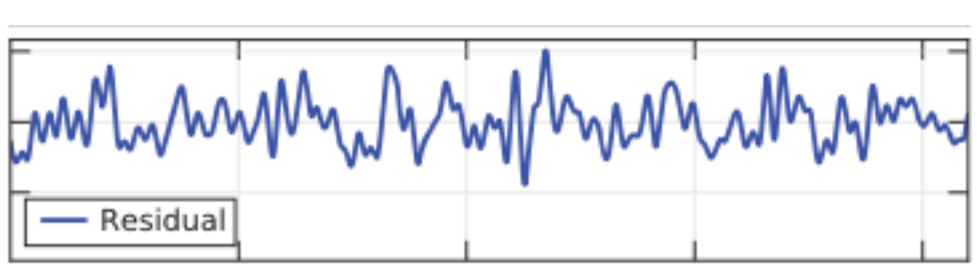


(Abbott et al. PRL 118 (2017) 221101)

- Subtracting **best-fit GR waveform** model (MaP) from data

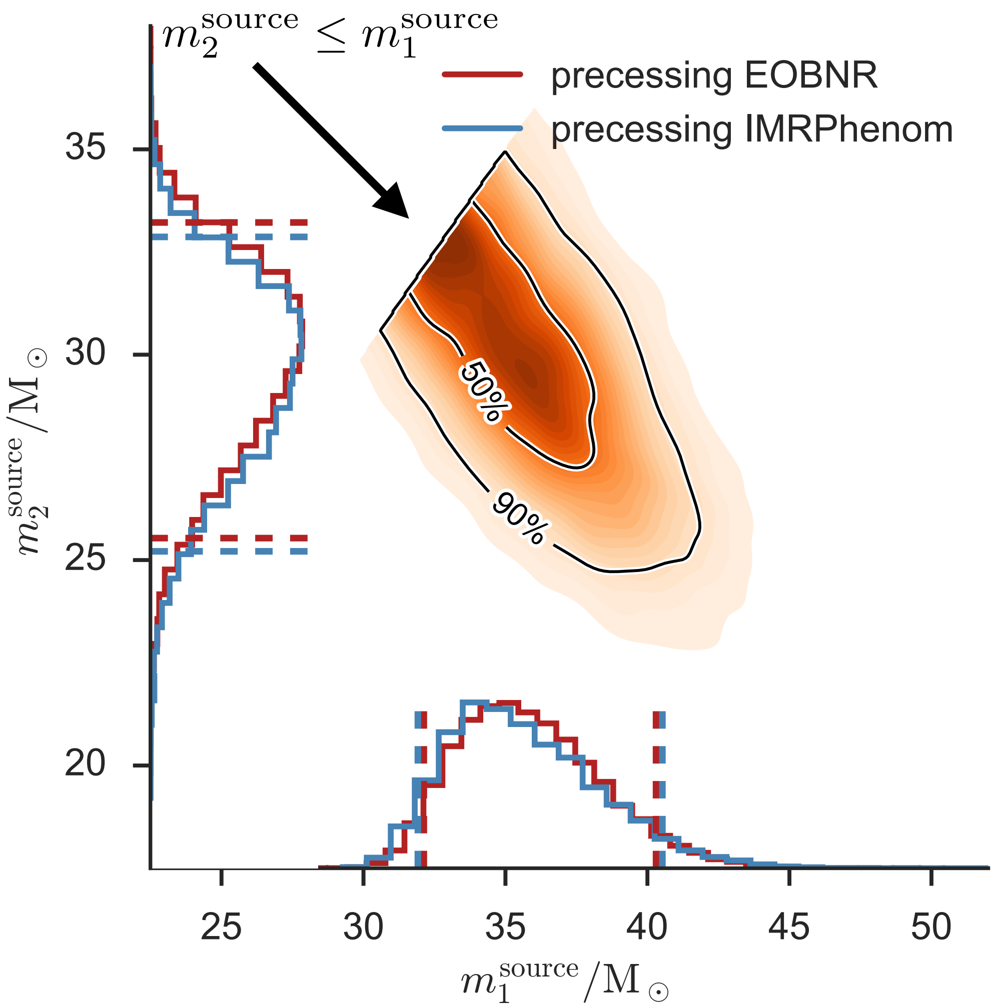




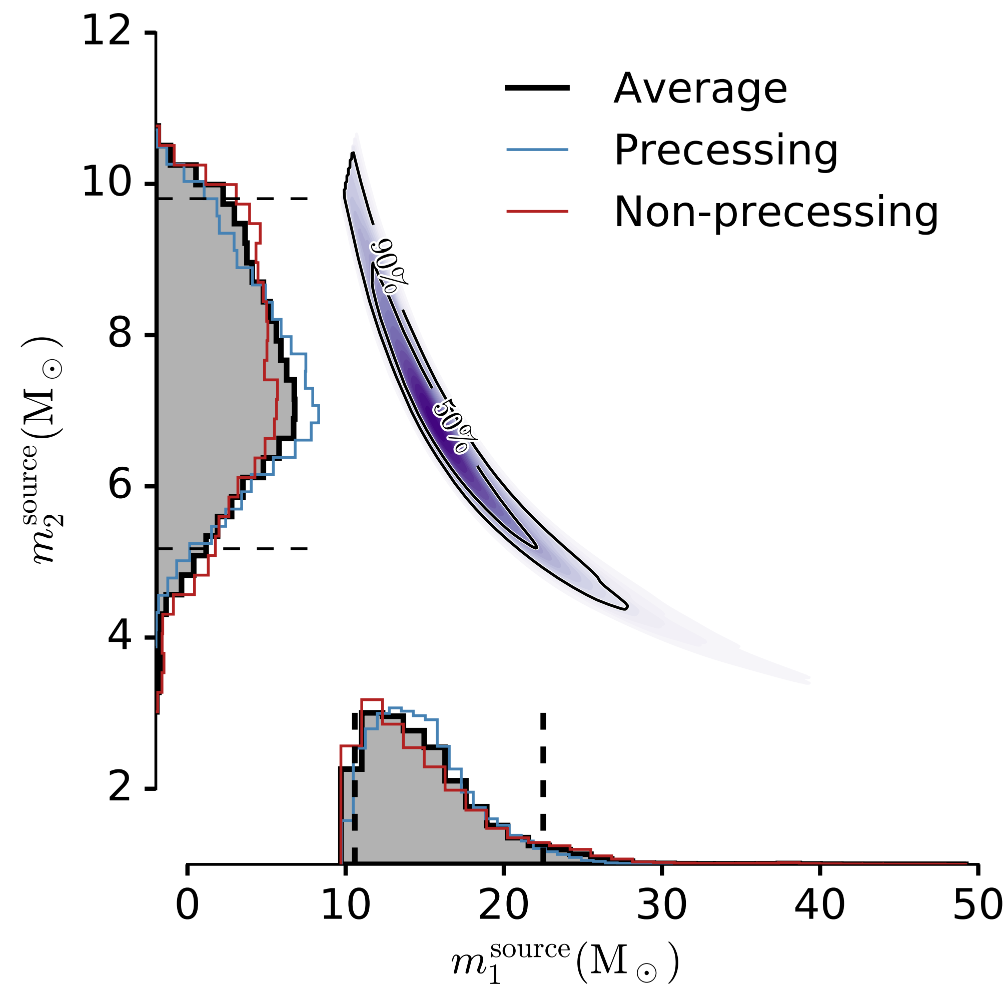
Unveiling binary black holes properties: masses

GW150914

(Abbott et al. PRX 6 (2016) 041014)



GW151226



(Abbott et al. PRL 116 (2016) 241103)

- We **measure best** the **“chirp” mass** $\mathcal{M} = M\,\nu^{3/5}$

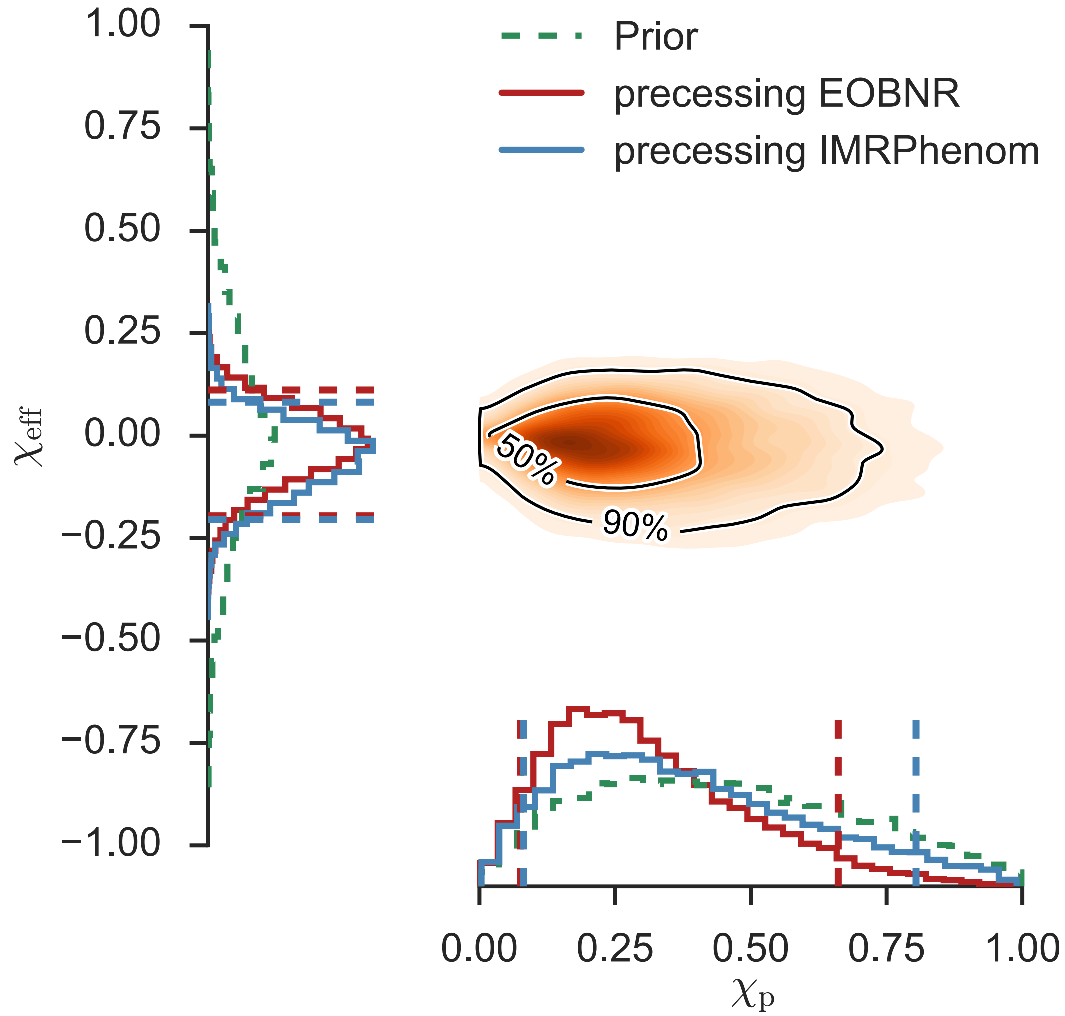
- **GW150914**: merger in band, total mass well measured, good measurement of individual masses.

- **GW151226**: merger outside band, individual masses measured less precisely.

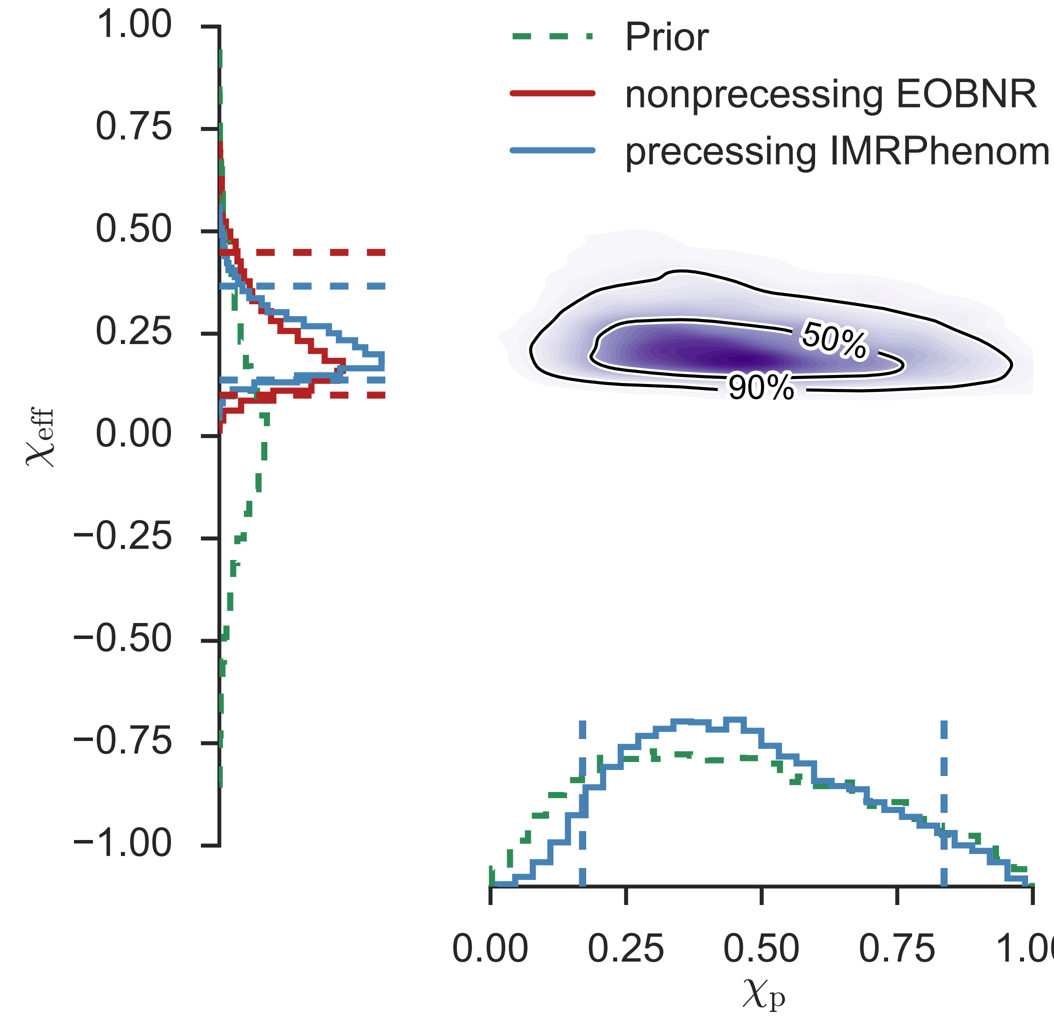
Unveiling binary black-holes properties: spins

(Abbott et al. PRX 6 (2016) 041014)

GW150914



GW151226



(Abbott et al. PRL 116 (2016) 241103)

- BHs' **spins not maximal**, and for GW151226 **one BH's spin larger than 0.2** at 99% confidence.

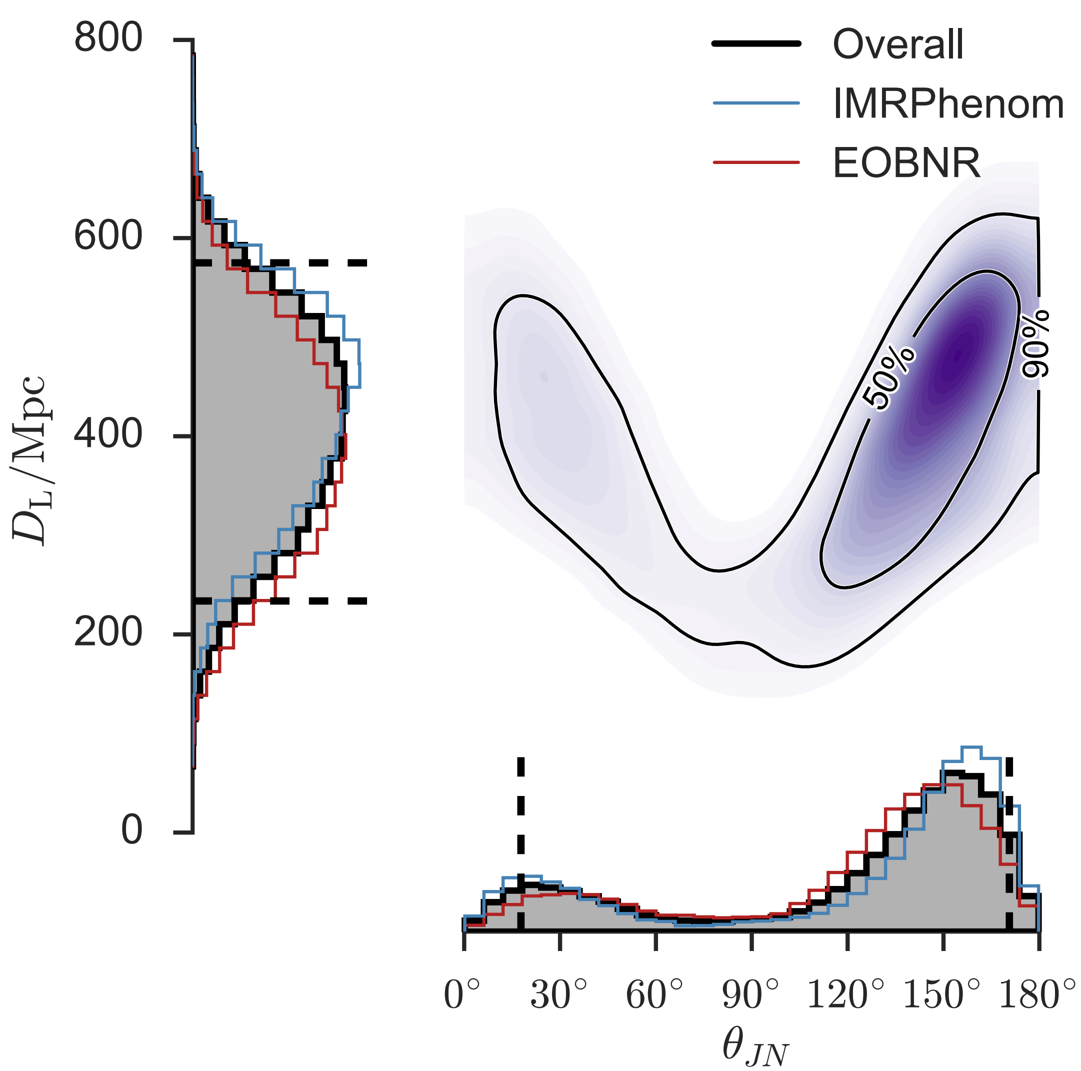
$$\chi_{\rm eff} = \left(\frac{\mathbf{S_1}}{m_1} + \frac{\mathbf{S_2}}{m_2}\right) \cdot \left(\frac{\hat{\mathbf{L}}}{M}\right)$$

- **Spin of primary BH < 0.7**. **No information** about **precession**.

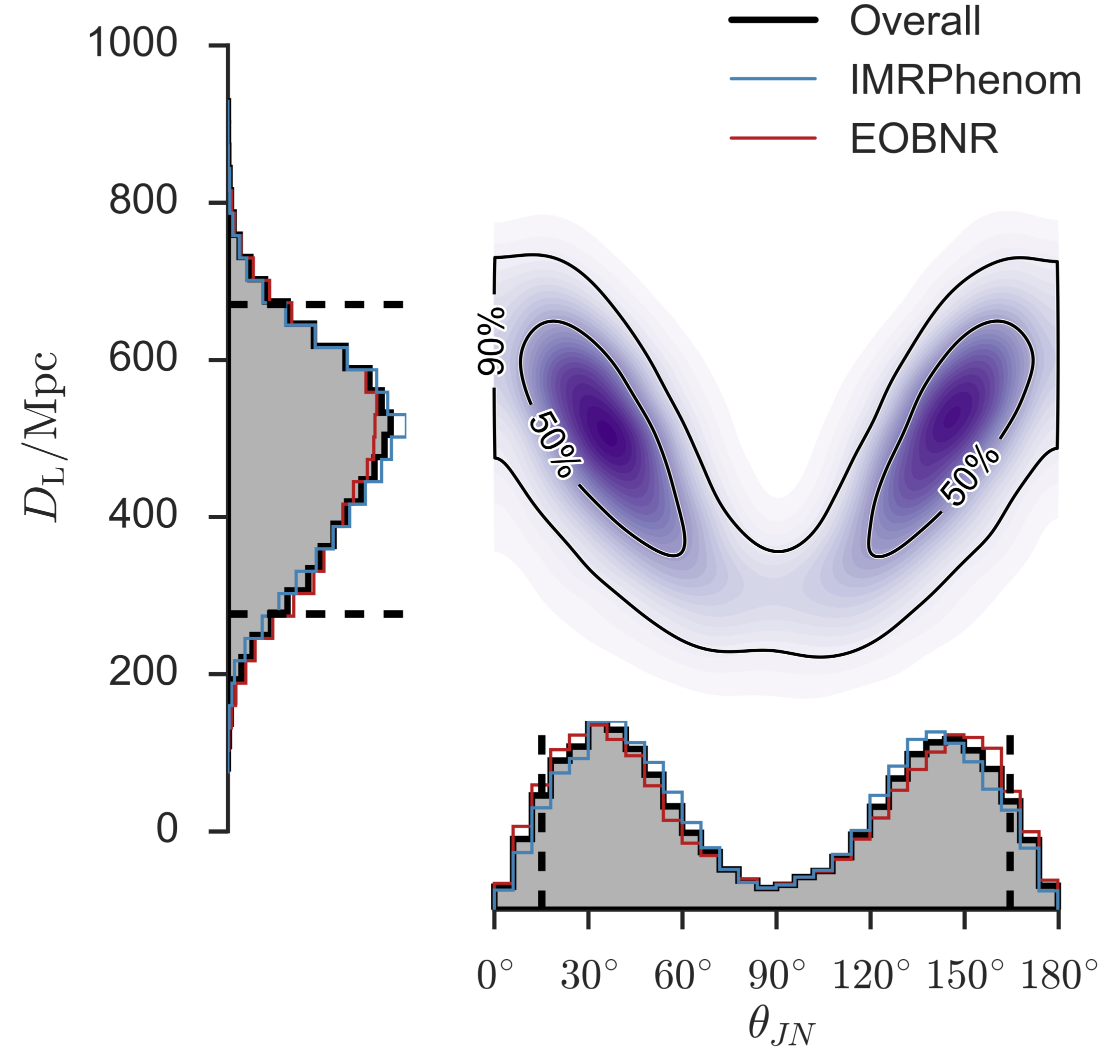
Binaries' distance and orbital-plane inclination

(Abbott et al. PRX 6 (2016) 041014)

GW150914



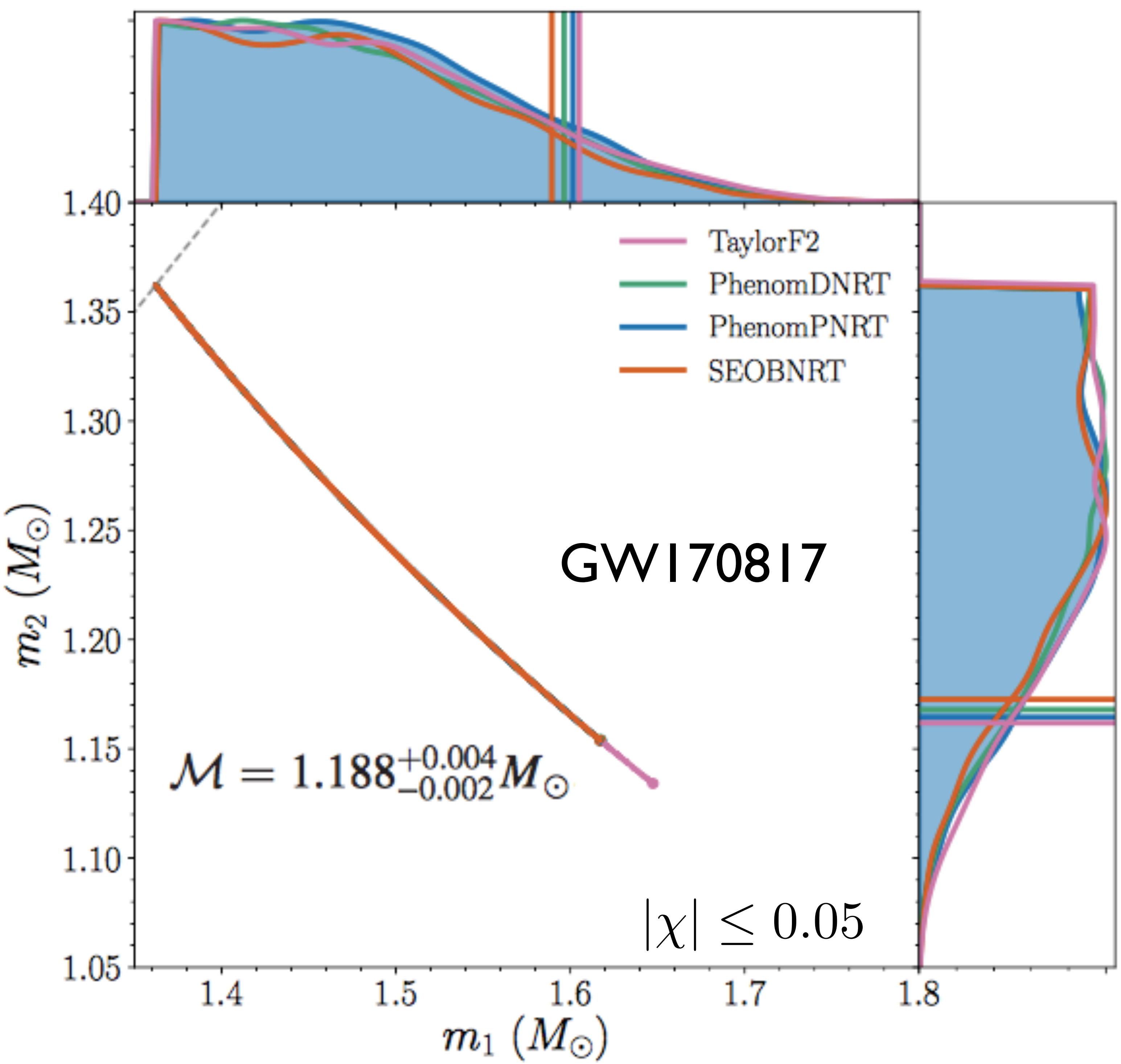
GW151226



(Abbott et al. PRL 116 (2016) 241103)

Unveiling binary neutron-star properties: masses

(Abbott et al. PRX 9 (2019) 1, 011001)



- **Degeneracy** between masses and spins.
- Observation of **binary pulsars** in our galaxy indicate spins are **not larger than ~0.04**.
- Current **measurements** of NS masses **dominated by statistical** error.

Bounding PN parameters: inspiral

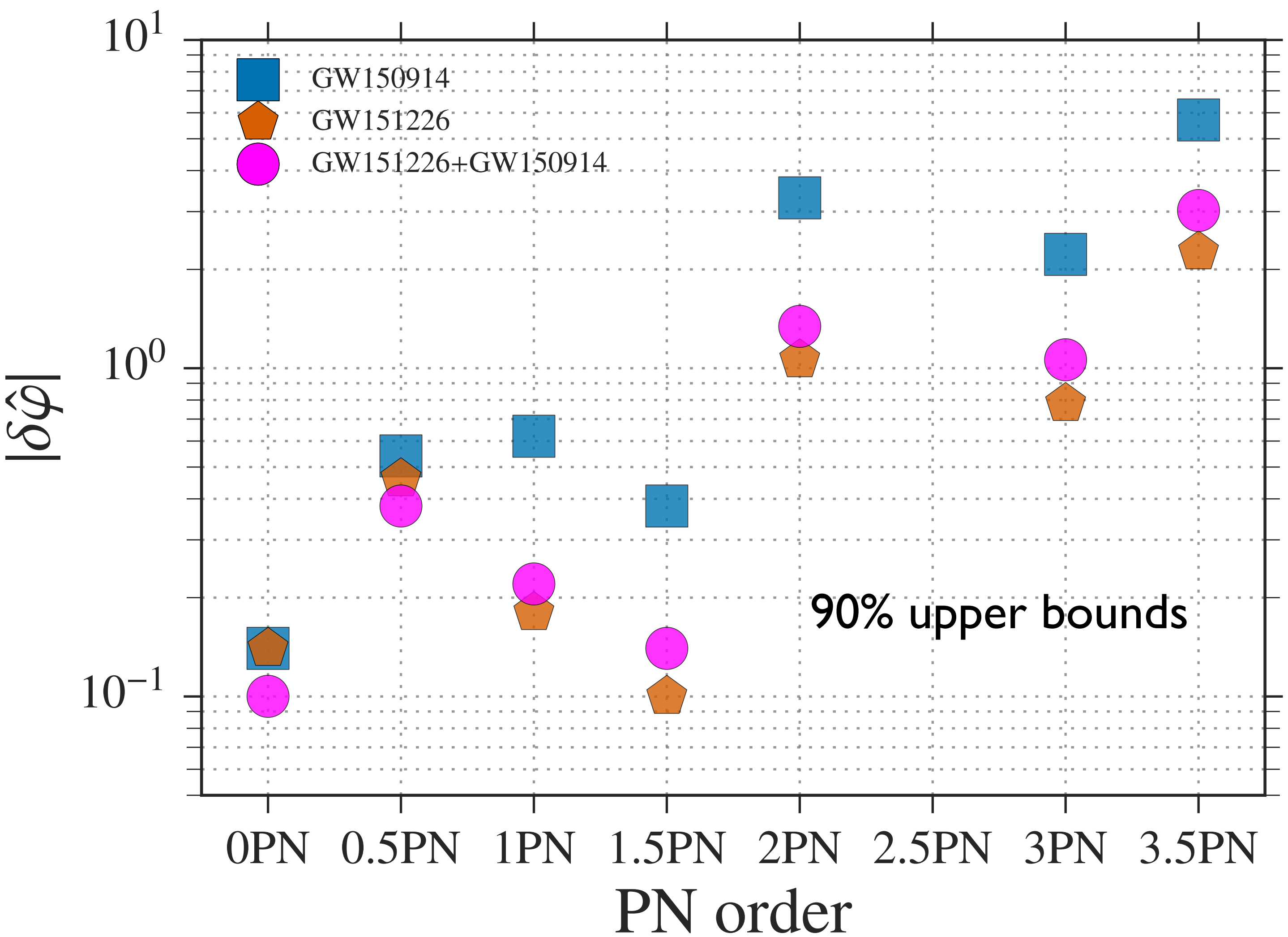
- GW150914/GW122615's **rapidly varying orbital periods** allow us to **bound higher-order PN coefficients** in gravitational phase.

$$\tilde{h}(f) = \mathcal{A}(f) e^{i\varphi(f)}$$

$$\varphi(f) = \varphi_{\rm ref} + 2\pi f t_{\rm ref} + \tilde{\varphi}_{\rm Newt} v^{-5} \left[1 + \tilde{\varphi}_{\rm 0.5PN}\, v + \tilde{\varphi}_{\rm 1PN}\, v^2 + \tilde{\varphi}_{\rm 1.5PN}\, v^3 + \cdots \right]$$

$$v = (2Mf)^{1/3}$$

(Abbott et al. PRX6 (2016))



- **PN parameters** describe: **tails** of radiation due to backscattering, **spin-orbit** and **spin-spin** couplings.

- **PN parameters** take **different values** in **theories** alternative to GR.

Bounding the graviton Compton wavelength (mass)

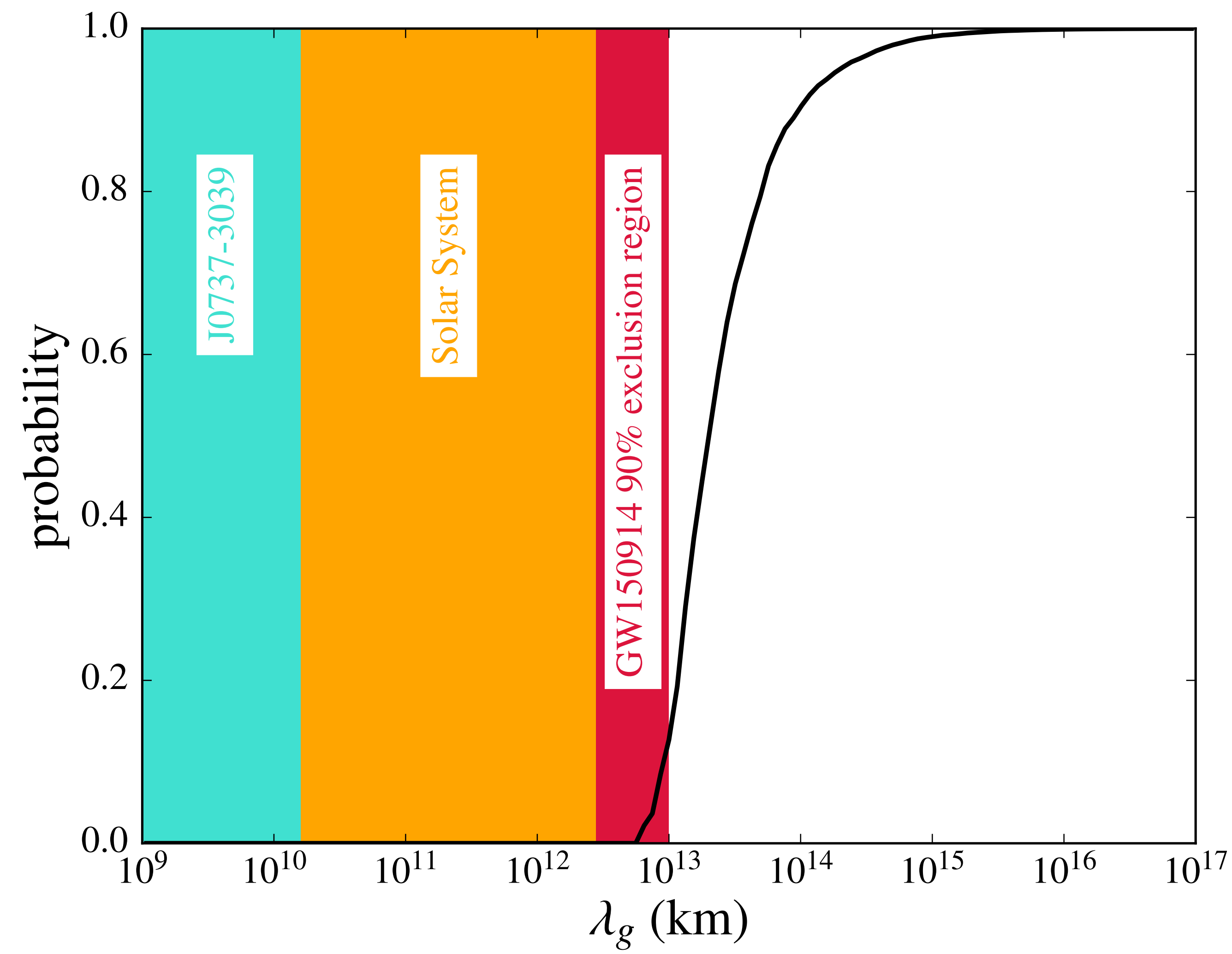
- Phenomenological approach: **modified dispersion relation**, thus GWs travel at speed different from speed of light.

$$E^2 = p^2c^2 + m_g^2 c^4 \qquad \lambda_g = \frac{h}{m_g c}$$

- **Lower** frequencies **propagate slower than higher** frequencies.

$$\frac{v_g^2}{c^2} = 1 - \frac{h^2c^2}{\lambda_g^2 E^2} \longrightarrow \varphi_{\mathrm{MG}} = -\frac{\pi D c}{\lambda_g^2 (1+z) f}$$

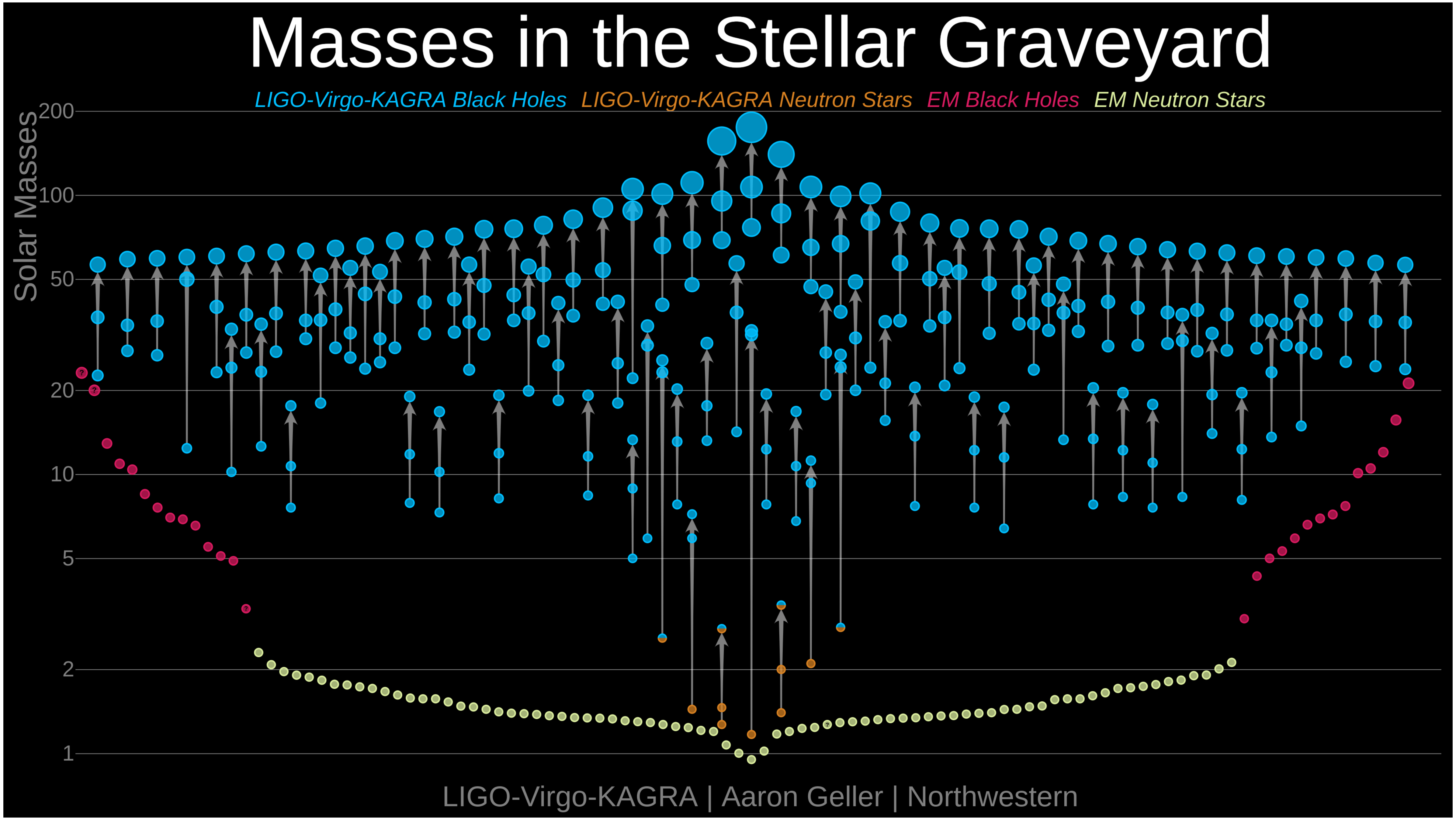
(Abbott et al. PRL 116 (2016) 221101)



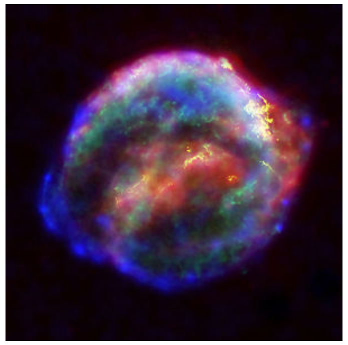
$$m_g \leq 1.2 \times 10^{-22} \mathrm{eV/c^2}$$

Black holes & neutron stars discovered so far by LIGOs/Virgo

- GWs from 90 **cosmic collisions!**



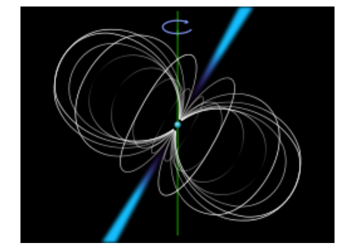
All kind of turmoil in the universe can ring spacetime



SN1604

- **Core of massive star** ceases to generate energy from nuclear fusion and **undergoes sudden collapse** forming a neutron star.
- **GW signal** is **unshaped burst** lasting for **tenths of millisecond.**

- Pulsars emit radio waves with extremely stable period.

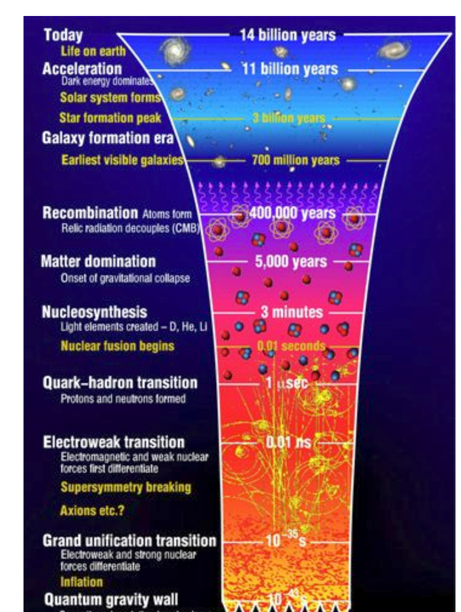


Crab Nebula

- **GW signal** is **continuous and periodic**.
- Best **constraint on NS "mountains"** from LIGO/Virgo **is 100 microns**!

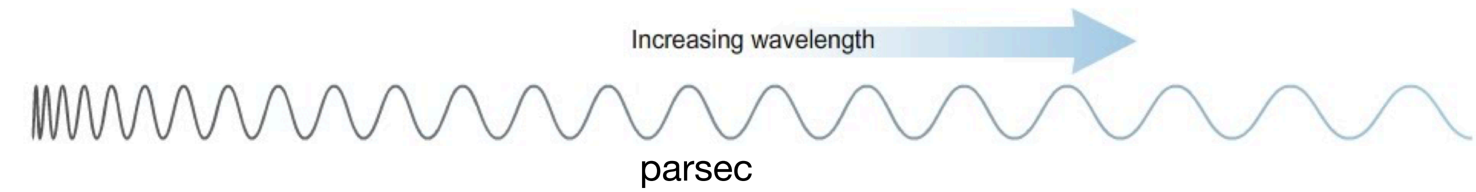
  (Abbott et al. ApJL 902 (2020) 1, L21)

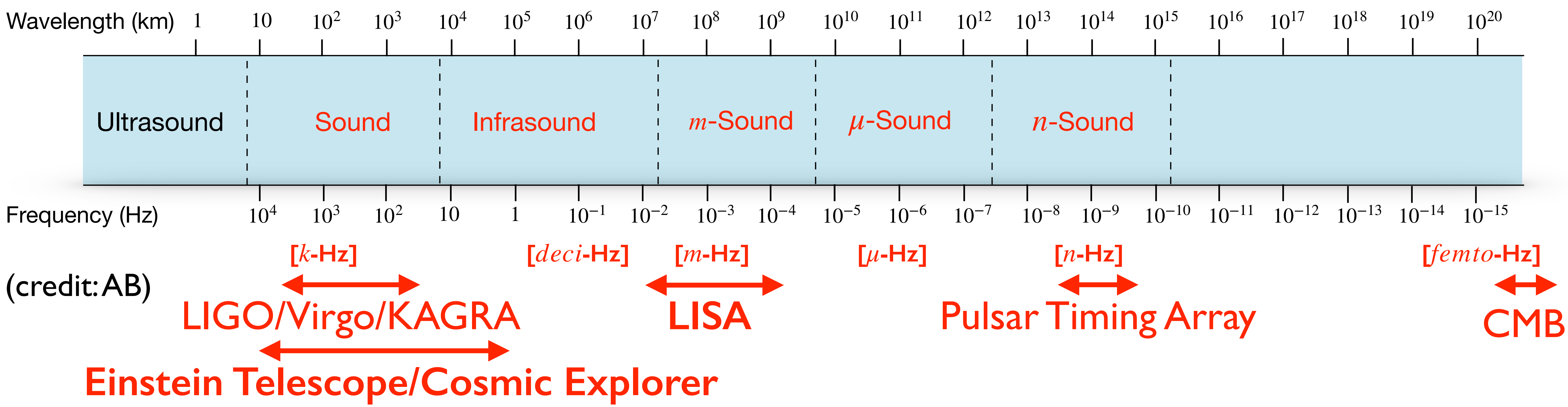
- **Peering back to the early moments** of our Universe.
- **Stochastic GW background produced** during rapid expansion of Universe **(cosmic inflation).**
- **GW signal like "random noise".**

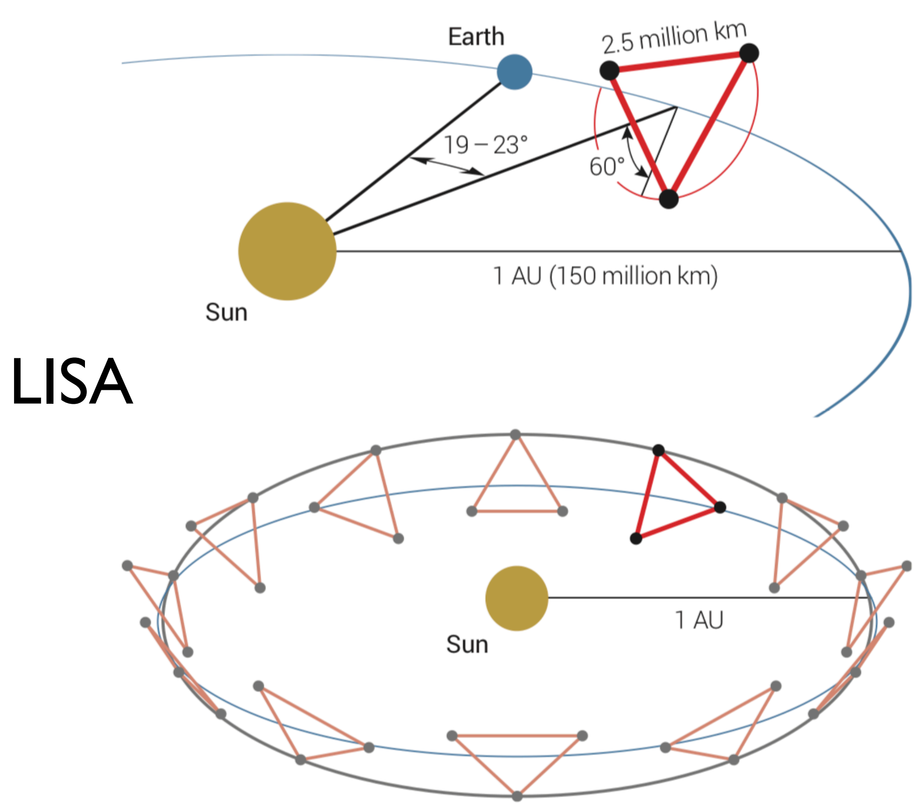


The bright future of GW astronomy in space and on the ground

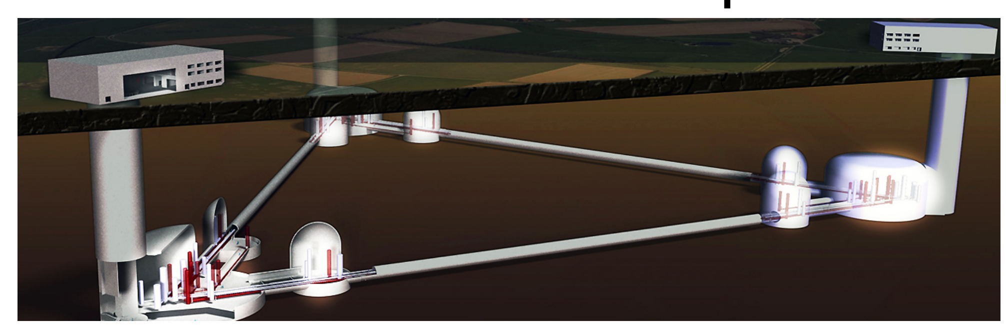
- **Gravitational-wave** spectrum







Einstein Telescope



Cosmic Explorer

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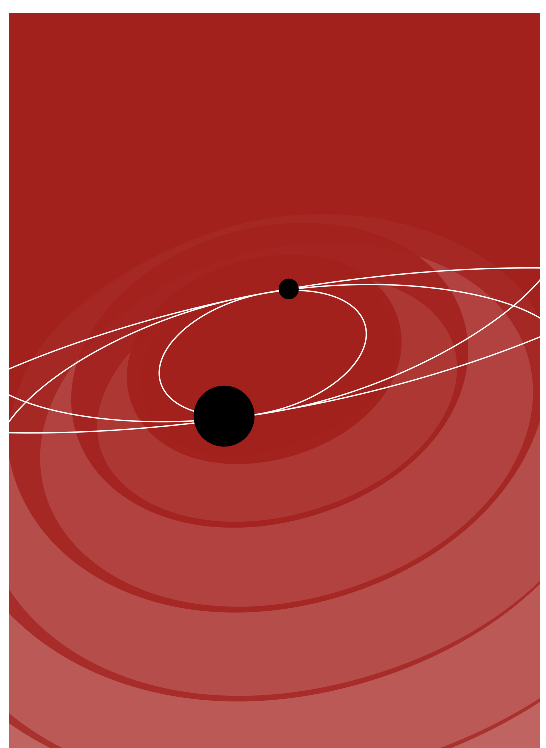


Thanks!

Hamilton-Jacobi formalism: "real" description

- The two-body **dynamics** can be **summarized in a coordinate-invariant manner** by evaluating the **"energy level" of the system**

(AB & Damour 1998)

$$H_{\rm real}(\mathbf{P},\mathbf{Q}) = H_{\rm Newt} + \frac{1}{c^2} H_{\rm 1PN} + \frac{1}{c^4} H_{\rm 2PN} + \cdots \qquad H^{\rm NR}_{\rm real} = H_{\rm real} - M c^2$$

$$\mathbf{P} = \frac{\partial S}{\partial \mathbf{Q}} \qquad R \equiv |\mathbf{Q}|$$

conservation of energy and angular momentum

$$S = -E^{\rm NR}_{\rm real}\, t + J_{\rm real}\, \varphi + S_{\rm R}(R, E^{\rm NR}_{\rm real}, J_{\rm real})$$

$$P_\varphi = J_{\rm real}\,, \quad P_{\rm R} = dS_{\rm R}/dR$$

$$H^{\rm NR}_{\rm real}(\mathbf{P},\mathbf{Q}) = E^{\rm NR}_{\rm real} \qquad \Rightarrow \quad S_{\rm R}(R, E^{\rm NR}_{\rm real}, J_{\rm real}) = \int dR \sqrt{\mathscr{R}(R, E^{\rm NR}_{\rm real}, J_{\rm real})}$$

@2PN it is a fifth-order polynomial in 1/R

radial action variable: $$I_{\rm R}(R, E^{\rm NR}_{\rm real}, J_{\rm real}) = \frac{2\alpha}{2\pi} \oint_{R_{\min}}^{R_{\max}} dR \sqrt{\mathscr{R}(R, E^{\rm NR}_{\rm real}, J_{\rm real})} \qquad \alpha = G\,\mu\, M$$

real "energy levels": $$E_{\rm real}(N_{\rm real}, J_{\rm real}) = M - \frac{1}{2}\frac{\mu\,\alpha^2}{N^2_{\rm real}}\left[1 + \mathcal{O}\left(\frac{1}{c^2}\right)\right] \qquad N_{\rm real} = I_{\rm R} + J_{\rm real}$$

Hamilton-Jacobi formalism: "effective" description

- We need to **associate to "real" two-body dynamics** $S_{\rm real}(z_1^\mu, z_2^\mu)$, an **"effective" one-body dynamics** in external spacetime with action:

(AB & Damour 1999)

$$S_{\rm eff}[z^\mu_{\rm eff}] = -\,\mu \int ds_{\rm eff} \qquad ds^2_{\rm eff} = -\,A_\nu(r)\,dt^2 + \frac{D_\nu(r)}{A_\nu(r)}\,dr^2 + r^2\,d\Omega^2$$

$$\mathbf{p} = \frac{\partial S^{\rm eff}}{\partial \mathbf{q}} \qquad r \equiv |\,\mathbf{q}\,| \qquad A_\nu(r) = \sum_{n=0}^{3} a_n(\nu)\left(\frac{GM}{r}\right)^n \qquad D_\nu(r) = \sum_{n=0}^{2} d_n(\nu)\left(\frac{GM}{r}\right)^n$$

$$p_\varphi = J_{\rm eff}\,, \qquad p_{\rm r} = dS^{\rm eff}_{\rm r}/dr \qquad S^{\rm eff} = -\,E^{\rm NR}_{\rm eff}\,t + J_{\rm eff}\,\varphi + S^{\rm eff}_{\rm r}(r, E^{\rm NR}_{\rm eff}, J_{\rm eff})$$

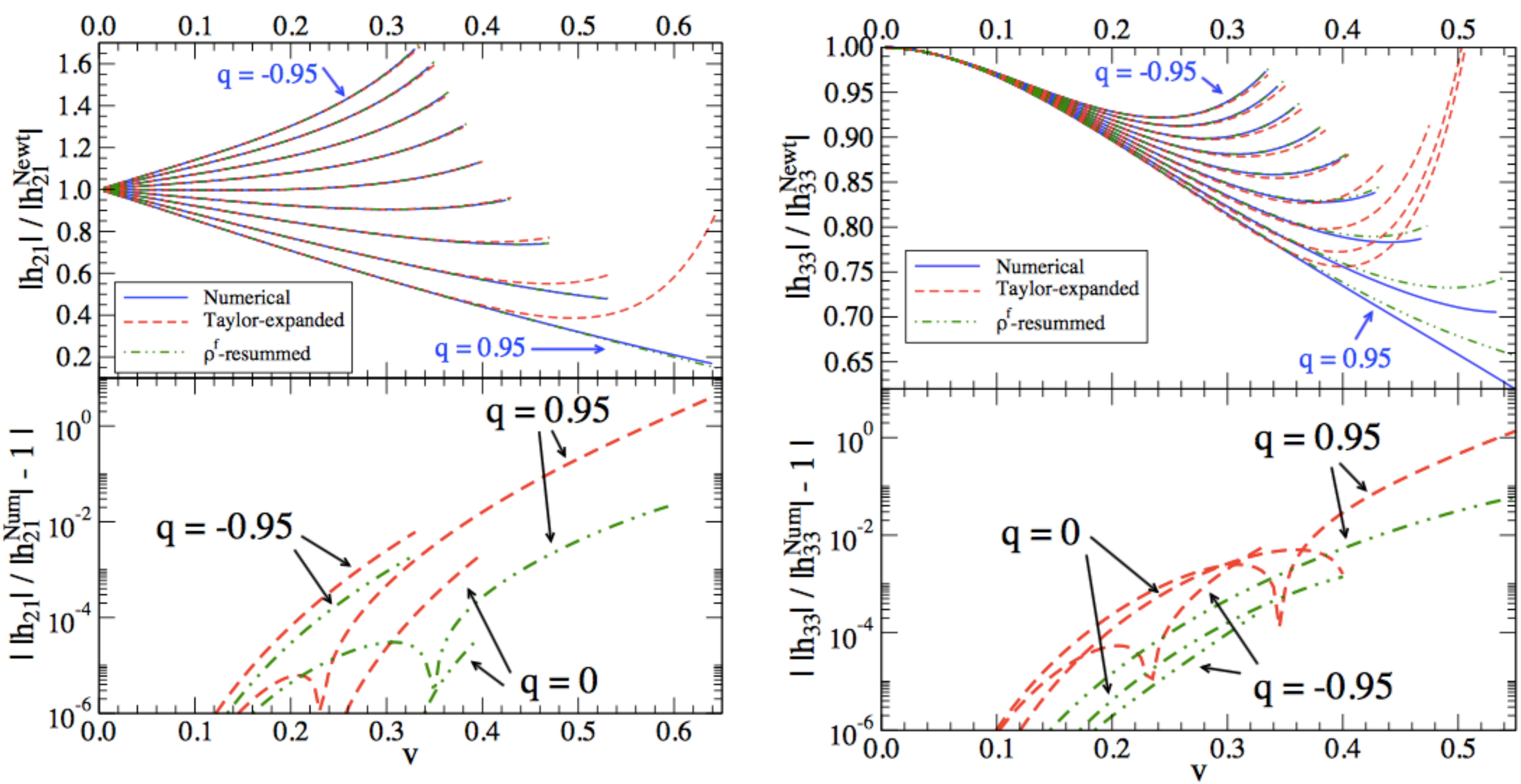
$$g^{\rm eff}_{\mu\nu}\,\frac{\partial S^{\rm eff}}{\partial x^\mu}\,\frac{\partial S^{\rm eff}}{\partial x^\nu} + \mu^2 = 0 \qquad \Rightarrow \quad S^{\rm eff}_{\rm r}(r, E^{\rm NR}_{\rm eff}, J_{\rm eff}) = \int dr \sqrt{\mathscr{R}^{\rm eff}(r, E^{\rm NR}_{\rm eff}, J_{\rm eff})}$$

$$\alpha = G\,\mu\,M$$

effective "energy levels": $E_{\rm eff}(N_{\rm eff}, J_{\rm eff}) = M - \dfrac{1}{2}\dfrac{\mu\,\alpha^2}{N^2_{\rm eff}}\left[1 + \mathcal{O}\left(\dfrac{1}{c^2}\right)\right]$ $\qquad N_{\rm eff} = I^{\rm eff}_{\rm r} + J_{\rm eff}$

EOB factorized modes with spins

- Gravitational **modes** for particle **orbiting Kerr BH** $q = a/M$



(Damour, Nagar & Iyer 09, Pan, AB, Fujita Racine & Tagoshi 10)

Completing EOB waveforms using NR/perturbation theory information

---

$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu) + a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

(Damour et al. 07-09, AB et al. 09, Pan et al. 09, Bernuzzi et al. 11, Pan et al. 11)

- **Once a_5 and a_6 are calibrated, the EOB light-ring (peak of orbital frequency) automatically occurs close to the time when h^{NR} reaches its peak**

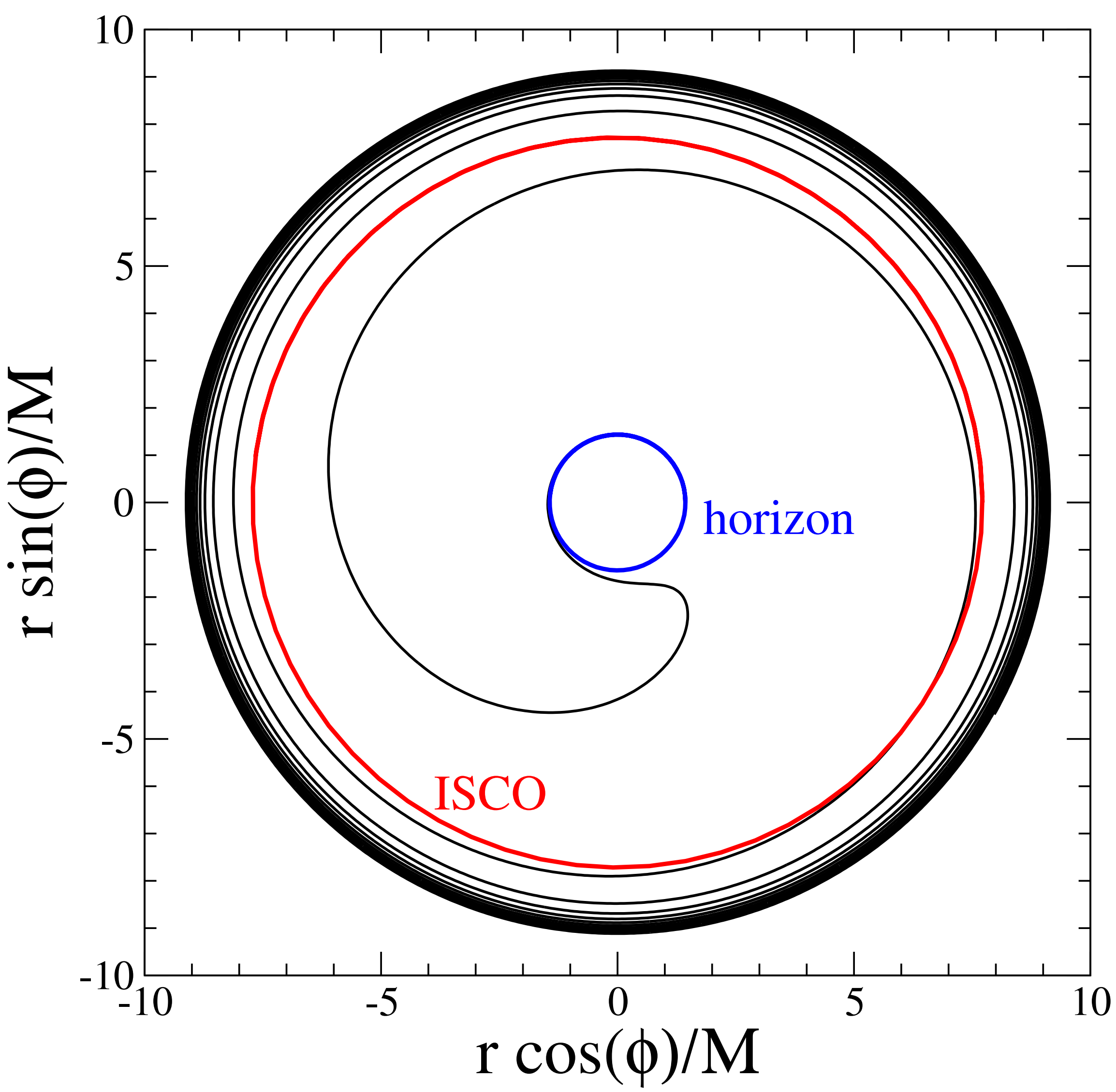
$$h^{\mathrm{NQC}} = \left[1 + \frac{p_{r*}^2}{(r\,\Omega)^2}\left(a_1 + a_2\frac{1}{r} + a_3\frac{1}{r^{3/2}}\right)\right] \exp\left[i\left(b_1\frac{p_{r*}}{r\,\Omega} + b_2\frac{p_{r*}^3}{r\,\Omega}\right)\right]$$

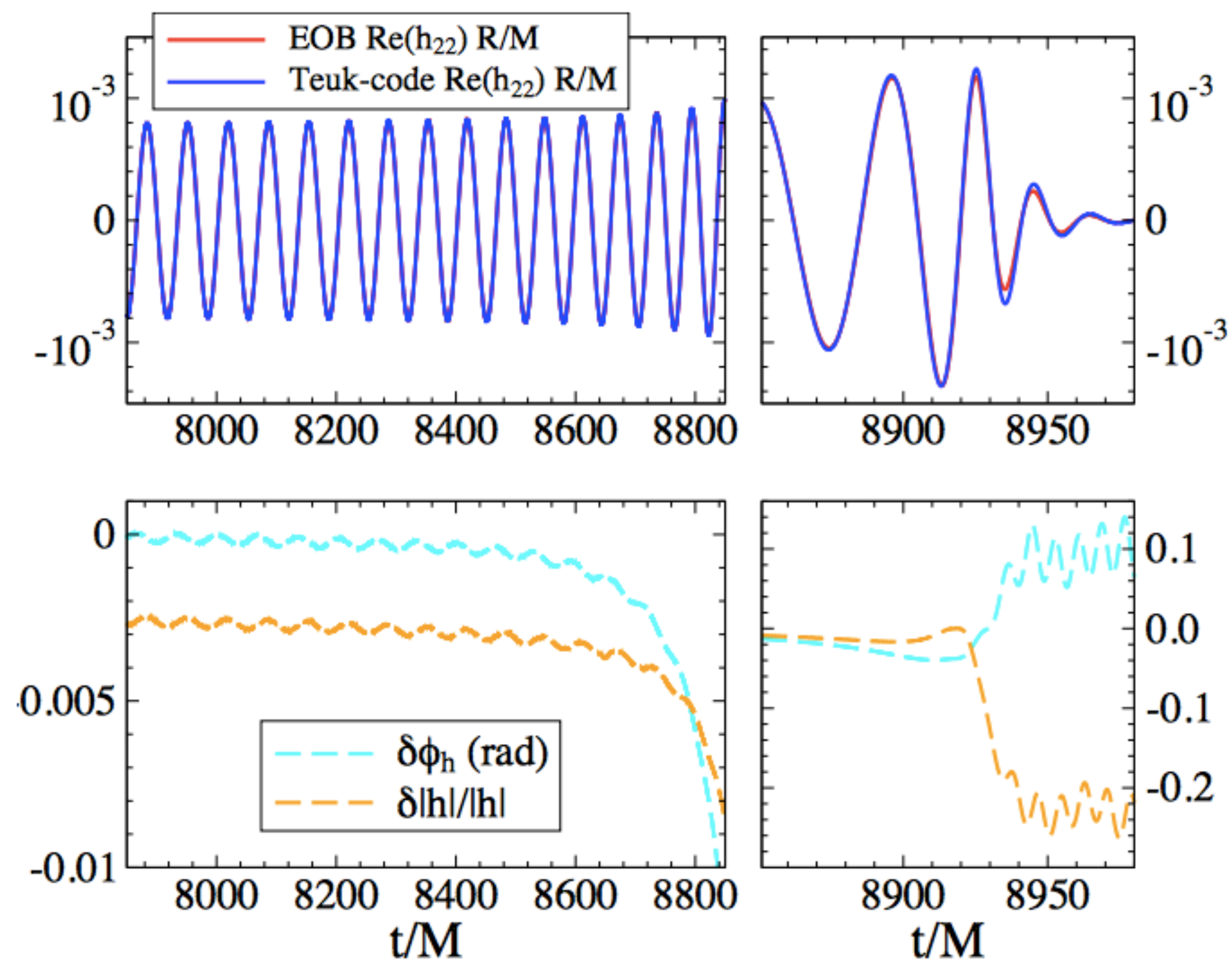
- **a_i, b_i are obtained imposing that the peak of h^{EOB} occurs at the EOB light-ring, its value and its second time derivative, ω^{EOB}, $\dot{\omega}^{\mathrm{EOB}}$, coincide with the NR ones**

$$|h^{\mathrm{NR}}(t^{\mathrm{peak}})|,\ |\ddot{h}^{\mathrm{NR}}(t^{\mathrm{peak}})|,\ \omega^{\mathrm{NR}}(t^{\mathrm{peak}}),\ \dot{\omega}^{\mathrm{NR}}(t^{\mathrm{peak}}) \Rightarrow \textbf{modeled as polynomials in } \nu$$

Calibrating EOB to Teukolsky-equation—based waveforms

- Solving **Teukolsky equation** for perturbations in **Kerr spacetime**





- **Retrograde** orbit: BH's spin = -0.5

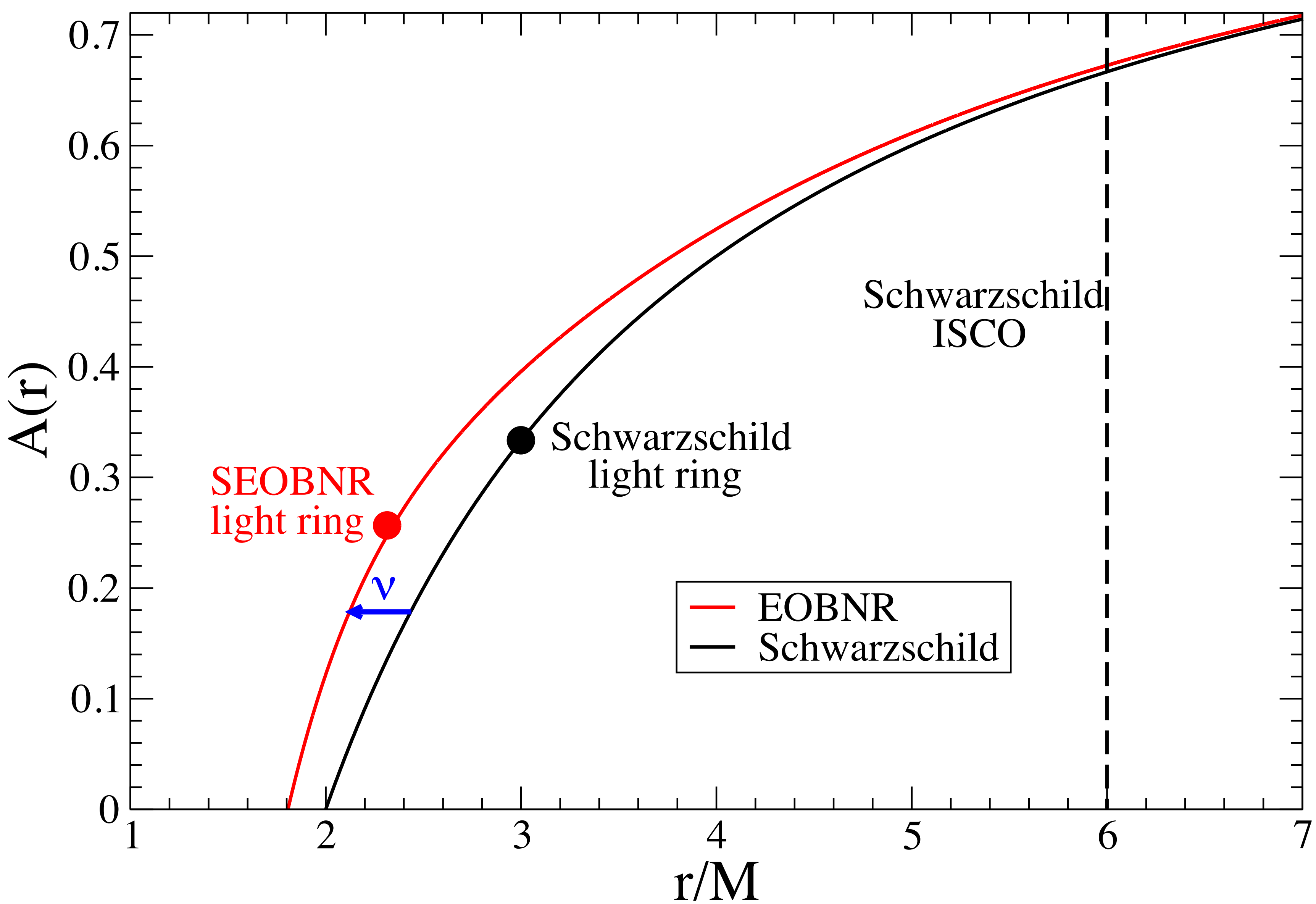
(Taracchini, AB, Khanna & Hughes 13, Barausse, AB, Hughes & Khanna 11)

(see also Damour & Nagar 07, *Bernuzzi et al. 10, 11, Harms et al. 14, 16)*

Strong-field effects in binary black holes included in EOB

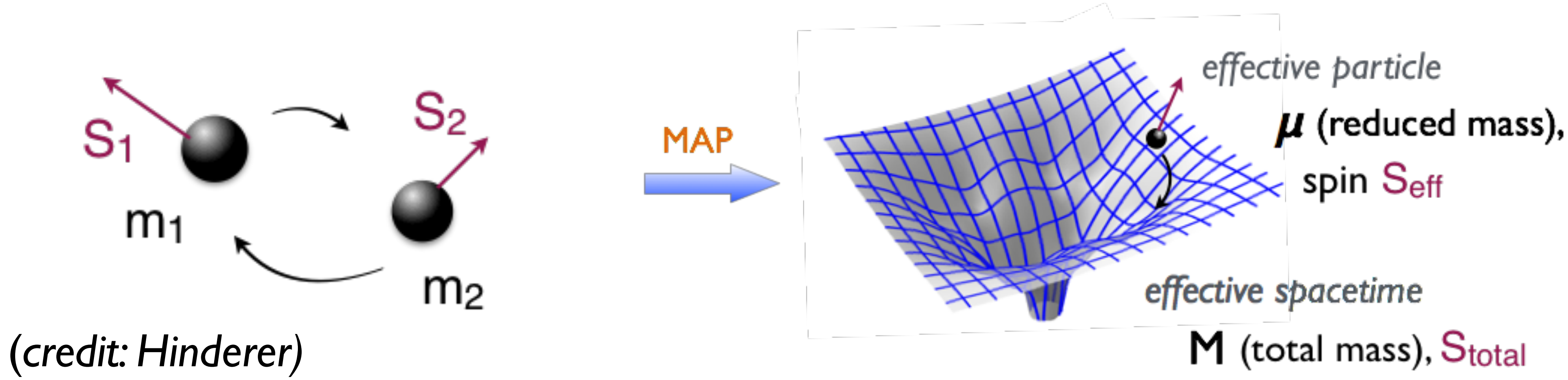
Finite mass-ratio effects make gravitational interaction less attractive

(Taracchini, AB, Pan, Hinderer & SXS 14)



$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu) + a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

EOB conservative spin resummed dynamics



$$H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}^{\nu}}{\mu}-1\right)}$$

- What is $H_{\rm eff}^{\nu}$ when compact objects carry spin?

$$g^{\mu\nu}_{\rm Kerr}\, p_\mu\, p_\nu = -\mu^2 \qquad H_0^{\rm Kerr} \equiv -p_0$$

(for simplicity we restrict to equatorial orbits)

- **Test spin** in Kerr spacetime: $H^{\rm Kerr} = \mu\sqrt{A^{\rm Kerr}\left(1+\frac{p_\varphi^2}{\mu^2 (r_c^{\rm Kerr})^2}+\frac{p_r^2}{\mu^2 B^{\rm Kerr}}\right)} + [G_{\rm S}^{\rm Kerr}(\mathbf{r})\, a + G_{\rm S_*}^{\rm Kerr}(\mathbf{r},\mathbf{p})\, a_*]\, p_\varphi$

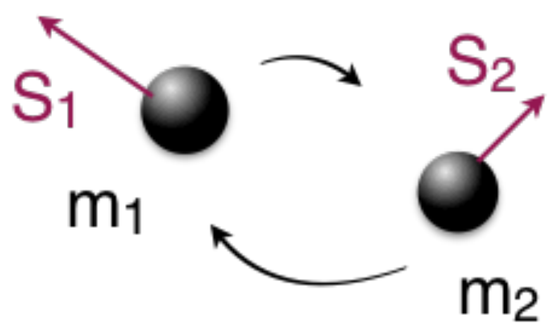
gyro-gravitomagnetic functions

$$a^* = \frac{S_*}{\mu M}$$

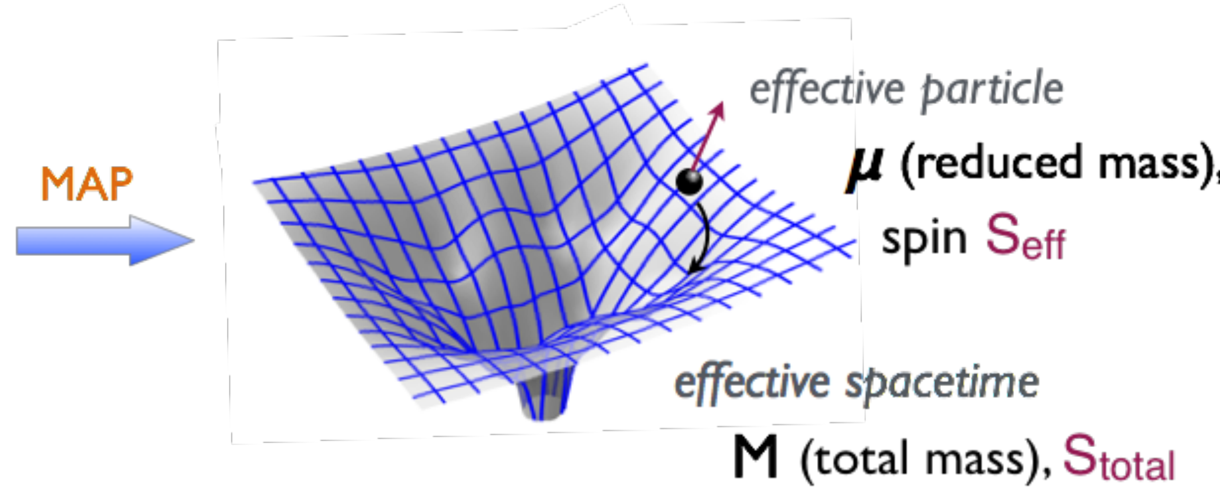
(Barausse & AB 11)

(see also Damour 01, Damour, Jaranowski & Schäfer 08; Damour & Nagar 14; Rettegno et al. 20)

EOB conservative spin resummed dynamics



(credit: Hinderer)



$$H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}^{\nu}}{\mu}-1\right)}$$

- What is $H_{\rm eff}^{\nu}$ when compact objects carry spin? Mapping is not unique, variants of $H_{\rm eff}^{\nu}$ exist.

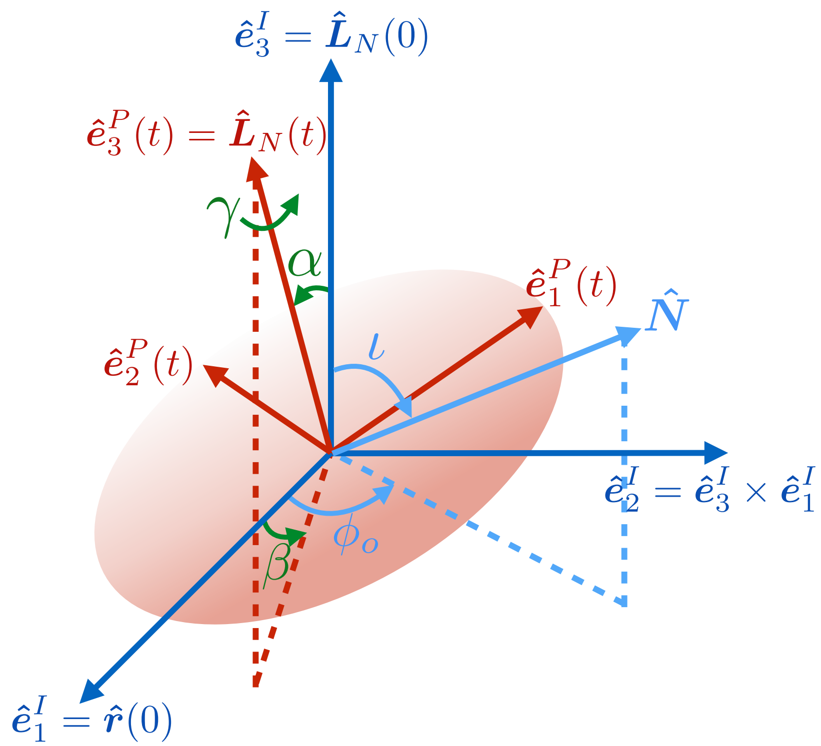
$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad \mathbf{S}^* = \frac{m_2}{m_1}\mathbf{S}_1 + \frac{m_1}{m_2}\mathbf{S}_2$$

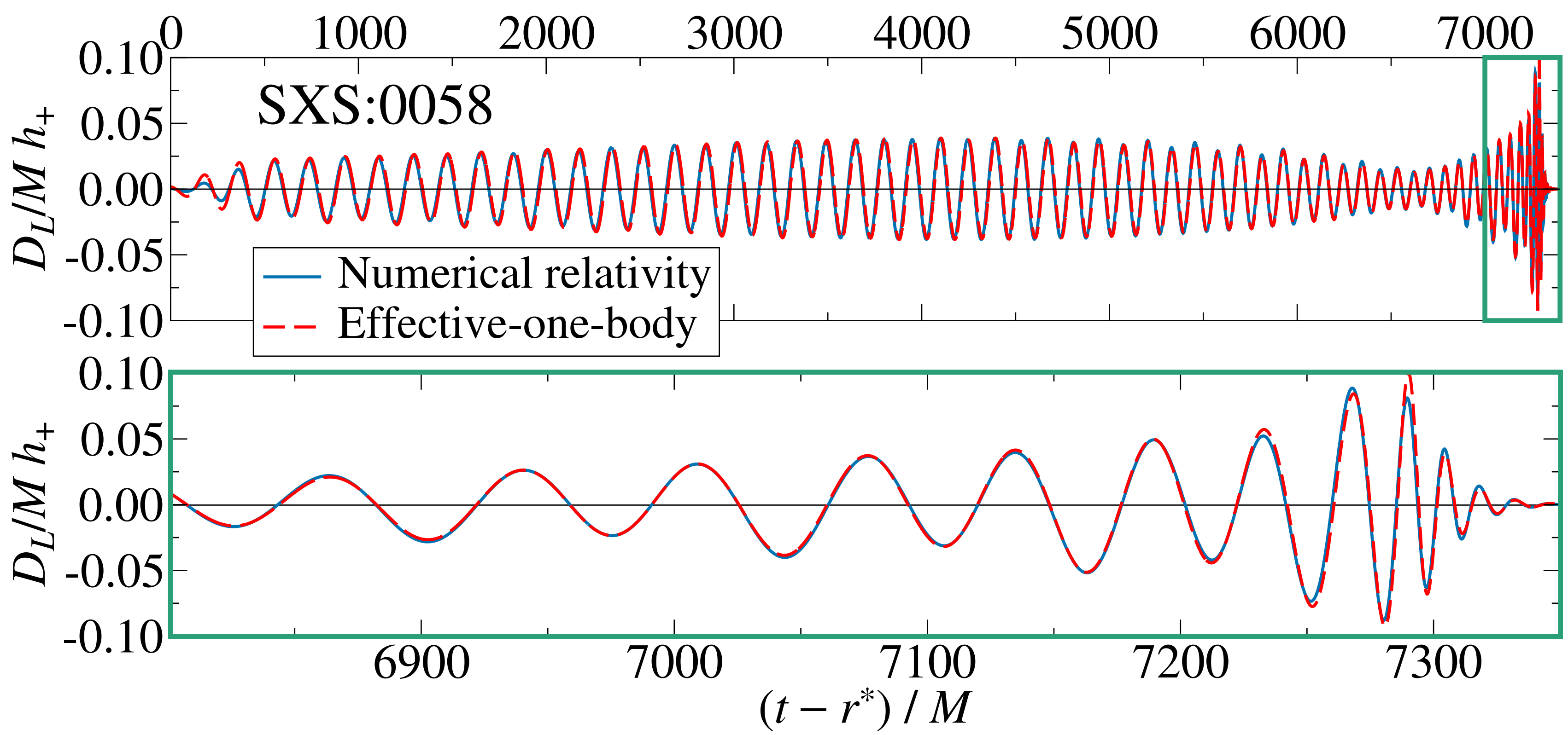
- **Test spin**: $$H_{\rm eff}^{\nu} = \mathbb{H}_{\rm eff}^{\text{Kerr-orb},\nu} + [g_{\rm S}^{\nu}(\mathbf{r},\mathbf{p})\,S + g_{\rm S^*}^{\nu}(\mathbf{r},\mathbf{p})\,S^*]\,p_\varphi + H_{\rm eff}^{\rm SS,\nu}$$

(Barausse, Racine & AB 10; Barausse & AB 11, 12; Vines et al. 16; Khalil et al. 20)

(see also Damour 01 Damour, Jaranowski & Schäfer 08; Damour & Nagar 14; Rettegno et al. 20)

Spinning precessing waveform models





Precessing (co-rotating) frame

(AB, Chen & Vallisneri 03, Boyle et al. 11, Schmidt et al. 11, O'Shaugheessy et al. 11)

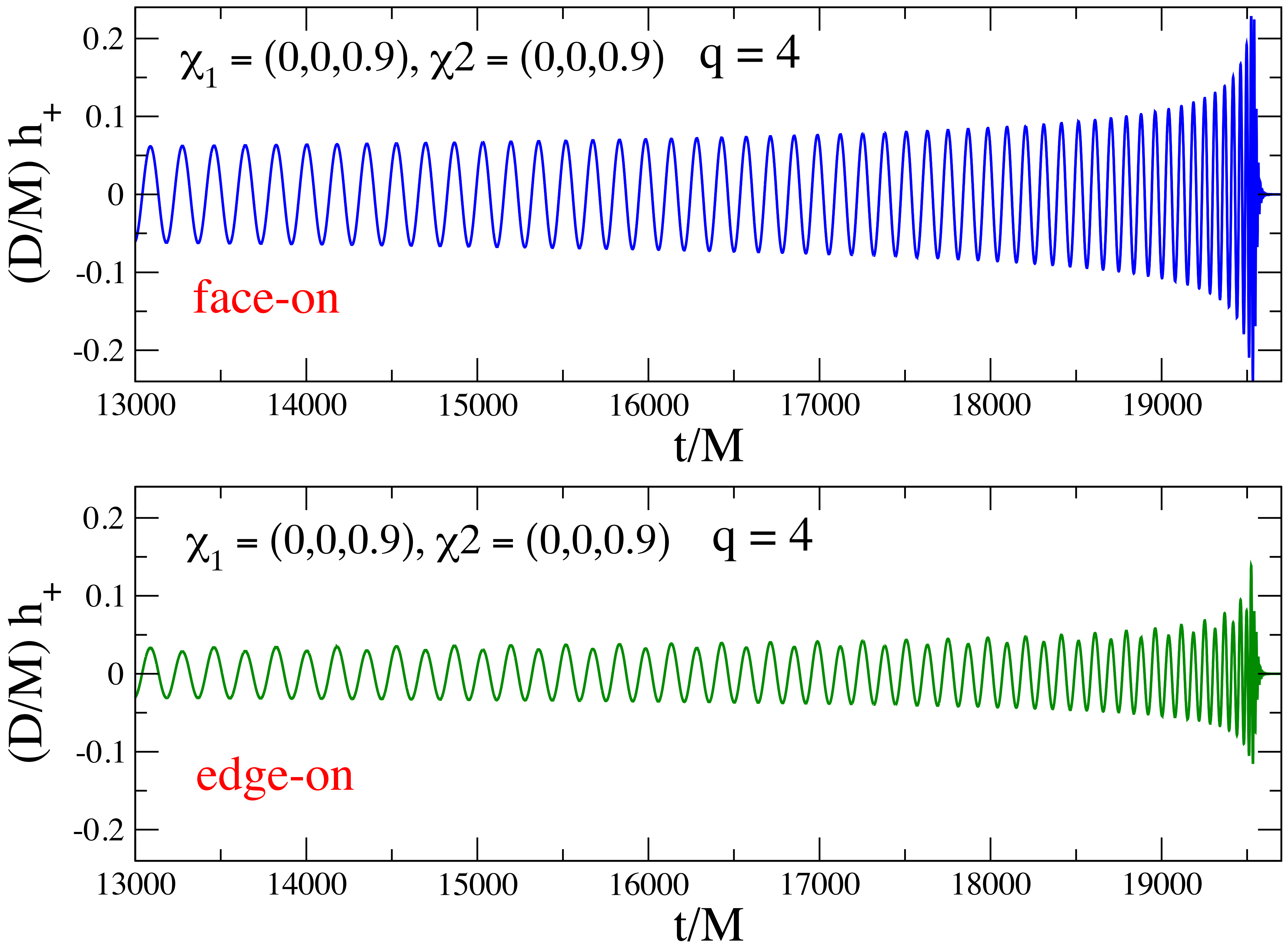
(Pan et al. 14, Babak et al. 16)

- **Single** effective-**spin precessing** waveform model in **frequency domain** (IMR phenomenological, 13-independent parameters). *(Schmidt et al. 12, Hannam et al. 14)*

- **Double-spin precessing** waveform model in **time domain** (EOBNR, 15-independent parameters). *(Pan et al. 14, Babak et al. 16)*

Effect of orientation of binary's orbital plane

spin nonprecessing binary



Effect of orientation of binary's orbital plane (contd.)

spin precessing binary